Emergent black hole physics from conformal field theory thermodynamics

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Emergent Black Hole Physics from Conformal Field Theory Thermodynamics

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Chapter 1

Introduction

Starting with our ancient ancestors, who stared up at the sky and discovered patterns in the great revolving mess of stars, and progressing to our contemporaries who are studying data from the large hadron collider and Planck telescope our approach to understanding and documenting the patterns of nature has improved significantly. All these observations, while beautiful in their own way, would be without greater meaning without strong theoretical models telling us how to interpret what we observe.

Starting with Newton's universal law of gravitation and evolving to Einstein's theory of general relativity our models for understanding the motion of objects on the celestial sphere has steadily matured. The power of these theories is emphasized by the strong terms they are respectively prefaced by; universal and general. They are believed to hold for all of the observable matter in the universe. And at least below the length scale of galaxies, where famously dark matter potentially takes effect, there exist no experimental counter-examples.

Concurrently at the level of small scales, atomic physics eventually gave way to nuclear physics and in turn to standard model physics as the smallest known description of nature. At the level of atomic physics quantum mechanics starts to play a role. This was noted as early as Bohr's original proposal that the energy spectrum of the hydrogen atom is quantized into discrete separations. The later discovery of the Lamb shift in the Lamb-Retherford experiment and its theoretical explanation further paved the way towards a quantum field theory as a model for particle physics.[4]

These quantum field theories provides us with the best description we currently have of most phenomena in nature at small scales. The standard model of particle physics is itself a quantum field theory and has been tested to great success in accelerator experiments. Arguably beyond levels that should realistically be expected. Beyond particle physics quantum field theory finds additional application in the mean field approach to condensed matter models and to pure mathematics such as in the construction of Calabi-Yau spaces [5]. As such there exists a lot of motivation to understand quantum field theory as clearly as possible. Both fortunately and unfortunately a general quantum field theory is an extremely complex mathematical system. The mentioned theoretical success is in large part due to the fact that in the quintessential historic field theory, quantum electrodynamics, the predicted field disturbances (electrons, positrons and photons) interact very weakly among each other at the energy regime where experiments are performed. Despite the overwhelming motivation the fundamental underlying mathematical basis for strongly interacting systems is not so well understood. The wide range of phenomena that strongly interacting systems can display distinctly emphasizes the complexity of these models. These systems can show examples of large classes of collective behavior such as: phase transitions, topological defects and thermalization. The required integrals that define the measurable observables for these phenomena are for all practical purposes unsolvable when there does not exist a way in which they are close to already known integrals.

The level of complexity of strongly-coupled quantum field theories is underlined by the holographic principle. A relatively well-controlled class of quantum field theories, conformal field theories, are believed to capture the full dynamics of gravity in a higher-dimensional space. This would already be greatly impressive if these models would capture the full non-linear dynamics of Einstein's theory of gravity, but the construction produced in the 1990s suggests that the full theory captures much more.

The first and most famous constructive example of such a duality is the anti-de Sitter/conformal field theory correspondence by Juan Maldacena. In this construction it was found that the Hilbert space of strongly coupled supersymmetric Yang-Mills theory at large number of colours N contains states equivalent to excitations of type IIB supergravity on the product of five-dimensional anti-de Sitter space and a five-dimensional sphere. This provides an explicit construction of the holographic principle proposed in [14], where on general principle the possibility of the formation of black holes proclaims that the fundamental dynamical quantum constituents of a quantum theory of gravity are constrained to live on the boundary of the universe. This is reminiscent to how a hologram can encode a full three-dimensional image onto a two-dimensional surface, hence the name holographic principle.

Furthermore it was conjectured that $\mathcal{N}=4$ SYM goes beyond the supergravity limit and captures the full theory of quantum gravity on $\mathrm{AdS}_5 \times S^5$ [8]. The property of $\mathcal{N}=4$ SYM to capture the full quantum theory of gravity at all gauge interaction strengths and orders of N is sometimes called the strong AdS/CFT conjecture [12]. This conjecture has had extremely far-reaching implications. Type IIB supergravity is constructed as a low-energy limit of type IIB string theory, while string theory is believed to be UV complete the conventional realm of string theory is restricted to first quantized strings. The strong AdS/CFT conjecture supposes that the full collective quantum dynamics of string field theory is contained within $\mathcal{N}=4$ SYM, since our conceptual control over strongly-coupled field theory is much stronger than on string field theory it has been proposed that we might think of the boundary theory as *defining* quantum gravity [8].

While Maldacena's discovery of the decoupling argument that resulted in the AdS/CFT duality was a watershed moment in modern theoretical physics, there had been some historic build-up towards such a discovery. A non-exhaustive list of historic precedents that slowly build up to AdS/CFT include:

- Preceding the proposed AdS_3/D1-D5 brane duality by one year, Strominger and Vafa concluded that counting up all the BPS states of the two-dimensional theory obtained by stacking one-dimensional and five-dimensional branes on top of each other adds up to the number of states of a black hole as counted by Bekenstein [16].

- Besides the holographic principle, AdS/CFT also provides a constructive string theory realisation of another idea noted by Gerard 't hooft in 1973 that a Yang-Mills theory where the number of colour fields is taken to be very large in some sense approaches a string theory [13, 15].

- In a more extreme example of dimensional reduction of a gravitational theory, the one-dimensional BFSS matrix model among other things contains a subsector of the states of eleven-dimensional M-theory [17].

- While the true power of AdS/CFT lies in the equivalence of dynamics, and not just kinematics, it was observed in 1986 by Brown and Henneaux that the charge algebra of asymptotic diffeomorphisms of AdS_3 gravity matches the Virasoro charge algebra of 2d conformal field theory [18].

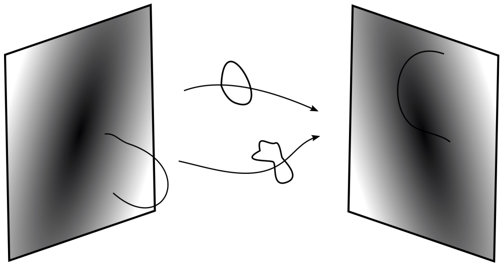


Figure 1.1: Two D-branes communicating by emitting and absorbing closed strings

While the topics covered in this manuscript will eschew the string theory origin of AdS/CFT it is useful to keep in mind for the sake intuition. For this reason it makes sense to make a small detour and revisit the original construction.

1.1 Holograms from Branes

String theory is more than just the dynamics of a handful of scattering strings. It also describes larger objects that arise as the collective behaviour of many strings, an example of these object are D-branes. These D-branes are typically portrayed as large sheets or higher dimensional objects that float through space and on which open strings can end, see figure (1.1). It is these branes that played a pivotal role in the original formulation of AdS/CFT.

The underlying idea is that when one considers a situation where there are many D3-branes brought close to one another that there are two equivalent ways of obtaining the low-energy regime. Consider the situation where there are N number of D3-branes in type IIB string theory, the most straightforward way to take the low-energy limit is to keep only the lowest energy regime of the open and closed string sector and possibly some interaction terms. The degrees of freedom of the closed string spectrum are contained within a 10d supergravity multiplet, concurrently the low-energy degrees of freedom of the open string spectrum are contained within a 4d $\mathcal{N} = 4$ vector multiplet in the adjoint representation of $SU(N)$.

The alternative route to the low-energy regime is the following. The D3-branes are bona fide charged objects that carry mass, they are believed to correspond to extremal black p-brane solutions of the low-energy supergravity regime [20]. Stacking multiple D3-branes together leads to a non-extremal black brane geometry in the supergravity regime. The black brane metric contains a redshift factor similar to the Schwarzschild geometry of general relativity, as a result of which there are two possible types of low-energy modes from the perspective of an observer at infinity, either we have the massless modes anywhere in 10d space, or we have any kind massive high-energy mode that originated near the black brane horizon and redshifted as it approached the observer at infinity [10]. So a natural approach is to investigate the region where these massive modes originated from. Taking the limit of an oberserver that approaches the horizon of the black brane shows that an observer close to the horizon experiences a curved geometry that is essentially $\text{AdS}_5 \times S^5$.

As a result of these observations there are two distinct models that describe the low-energy excitations of a system containing N D3-branes, on the one hand we have

$$4d\ \mathcal{N} = 4 \text{ SYM } \times \text{ massless sector of 10d IIB sugra}$$

on the other hand we have

$$\text{Full IIB sugra on } \text{AdS}_5 \times S^5 \ \times \text{ massless sector of 10d IIB sugra}$$

It was found in [8] that there exists an appropriate decoupling limit of the string coupling constant α' such that in both cases the massless sector of 10d supergravity decouples from the other factor. Resulting in the identification of four dimensional $\mathcal{N} = 4$ SYM with IIB supergravity on $\text{AdS}_5 \times S^5$. In that same paper it was conjectured that the resulting duality of the two theories perseveres when one leaves the decoupling regime, this is what is now called the AdS/CFT conjecture.

While this construction motivates the existence of a duality between conformal field theories and gravity theories it does not provide a prescription how to compare processes on both sides of the duality. This gap is filled by the GKPW prescription or Witten prescription [9, 11]. This prescription states that local operators in the conformal field theory function as sources for supergravity fields at the boundary of the gravity theory. To put this quantitatively a d-dimensional CFT correlator of local operators as a function of the locations should match the function obtained if one computes the $d+1$-dimensional supergravity path integral where the action includes sources located at the boundary corresponding to the CFT insertions. This linking between sources and supergravity fields provides part of a ‘dictionary’ linking CFT operators to supergravity fields, for one since scalar fields have scalar sources we derive that scalar CFT operators correspond to scalar supergravity fields. Secondly it is well-known that infinitessimally perturbing the background geometry of a field theory induces a shift in the action proportional to stress tensor contracted with the metric perturbation. This clarifies an important part of the AdS/CFT dictionary

$$\text{CFT stress tensor} \longleftrightarrow \text{graviton field.}$$

Hence correlators of the stress tensor in the CFT correspond to graviton scattering amplitudes in the bulk, and CFT processes where the intermediate channel contains the stress tensor correspond to bulk interactions mediated through gravity. Furthermore this provides a deep insight that the flow and dynamics of the background geometry in the bulk gravity theory is deeply connected to the flow of energy in the CFT and hence its thermodynamics.

1.2 Black holes and Thermodynamics

Some of the more qualitative aspects of holography did not originate with AdS/CFT though. The link between gravity and thermodynamics preceded the AdS/CFT correspondence by decades. Hawking's area theorem of black holes states that in any classical process satisfying an energy conjecture the black hole area size never decreases [21]. This statements shares similarities with the second law of thermodynamics, it was additionally found that the other laws of thermodynamics have similar black hole analogues.

This connection between the laws of thermodynamics and those of black holes was first considered as more than just an analogy by Jacob Bekenstein in 1971. Bekenstein was led to question the entropy of black holes by his Ph.D. advisor John Wheeler, who asked him what happens to a cup of tea that falls into a black hole [22, 30]. His conclusion was that black holes themselves are objects that carry entropy and this entropy is proportional to the area of the event horizon. This is quantified in the now famous expression

$$S_{BH} = \frac{A}{4G}, \tag{1.1}$$

where G is Newton's constant. The suggestion that black holes carry entropy did not immediately gain traction within the physics community. One famous objection was that as objects that carry both energy and entropy it is unavoidable that black holes as objects also carry temperature. Hawking, in an attempt to disprove that black holes possess a temperature, came to

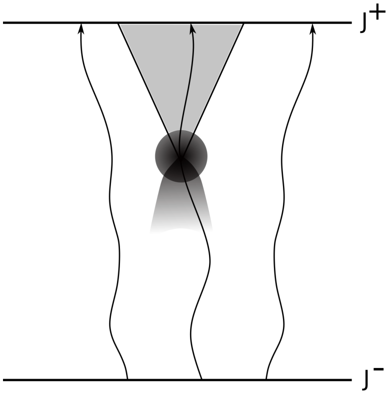


Figure 1.2: Hawking's argument in short: given a quantum field living on a background geometry corresponding to a collapsing star. The state of this quantum field on an initial slice in the infinite past J^+ will propagate to a part that falls into the black hole and a part that reaches infinity J^-. Hence an observer at infinity will only receive limited information of the initial state, and will therefore generally observe a mixed state. If the initial state was the vacuum the future mixed state will be a thermal state with a temperature fixed by the surface gravity of the post-collapse black hole [7].

the conclusion that this observation is in fact correct. The argument is based on quantum field theory in curved space, see figure 1.2. Given a quantum field state on any initial slice of time on background undergoing gravitational collapse. As the field propagates through space part of the field will propagate into the formed black hole whereas the rest will not and will ultimately reach an observer at infinity. Since part of the state fell into the black hole the observer will at most only possess partial information of the initial state. This missing of information manifests itself as the observer detecting a mixed state, if the initial state of the quantum field was the vacuum state then the late-time mixed state will be a thermal state whose temperature is fixed by the surface gravity of the black hole [7].

The resulting Hawking temperature of black holes is found to be consistent with the relationship between mass and entropy obtained by Bekenstein. Hence it is commonly observed that the subscript BH in (1.1) can either be read as short for 'black hole' or 'Bekenstein-Hawking'.

In the process of emitting thermal Hawking radiation black holes lose energy and hence shed their mass. In fact in asymptotic flat space the Hawking temperature is inversely proportional to the black hole mass. Meaning that as the black hole shrinks its temperature goes up and the rate at which it loses mass accelerates. This ultimately culminates in a violent explosion in the final stages of the 'evaporation process' [25]. Clearly this process is not within thermodynamical equilibrium.

In order to study the connection between black holes and thermodynamics it is much easier to study them within a context where they can achieve thermodynamical equilibrium. One way to achieve this is by placing the black hole 'in a box', with reflecting boundary conditions. If the black hole mass is tuned properly with respect to the size of the box then the black hole will reabsorb the emitted Hawking radiation before evaporating and eventually reach an equilibrium between emitted and absorbed radiation. Conveniently, though not coincidentally, if a black hole is embedded within anti-de Sitter space the ambient AdS space will essentially act as such a box. The stretching of AdS as we approach the boundary causes every outwards ballistic trajectory to eventually turn back inwards and fall back into the black hole. Even light is susceptible to this gravitational barrier and will eventually bounce back, but only after having actually reached the boundary of AdS. Hence an observer at infinity will experience a stable heat bath with a temperature given by the Hawking temperature of the black hole. If one imagines the boundary CFT as literally living on the boundary it is natural to state that the holographic dual of a stable black hole is a CFT thermal state.

As was laid bare by Hawking [26], the evaporation process of black holes reveals serious gaps in our understanding of quantum gravity. While it is tempting to think that any ultra-high energy scattering experiment falls deep within the realm of quantum gravity, they can be accessed by semi-classical gravity through clever use of reference frames [27]. In this same vein it is not so unreasonable to think of a high-energy scattering experiment resulting in black hole formation as an intermediate step. After waiting long enough for the resulting black hole to have evaporated all that remains is a mixed state corresponding to the thermal radiation. This proves to be a catastrophe for conventional quantum mechanics as this thought experiment seems to violate the unitarity of the S-matrix. Alternatively phrased the black hole appears to have deleted the information of the process that led to its existence, this is what is since called the information paradox.

AdS-Schwarzschild black holes (or BTZ black holes in 3d) are not exactly the same as their flat space counterparts, for one their temperature is proportional to the black hole mass as opposed to inversely proportional. Still they posses a radiating event horizon and an entropy that satisfies the Bekenstein-Hawking entropy formula, and they can even be formed by dynamically through collapsing matter [28]. Due to the eternal nature of AdS-Schwarzschild the information paradox has to be rephrased somewhat. An initial Hawking pair of quanta created near the horizon will be maximally entangled. One of the two of the pairs will escape to infinity as early Hawking radiation. The other Hawking quantum will scramble and after a period of time often called scrambling time will slowly be released from the black hole in terms of late Hawking radiation. This implies that the late time Hawking radiation is entangled with the early Hawking radiation which is incompatible with the conclusion that the radiation spectrum emerging from the black hole is thermal and hence uncorrelated. A large part of the power of AdS/CFT is that it rephrases these kinds of quantum gravity questions in terms of QFT questions. The related QFT problems involving the statistical mechanics of strongly interacting quantum fields in equally technically complicated and the exact dictionary relating CFT to gravity in AdS is not fully worked out [29], but it is much more clear on a conceptual level how to phrase the problem quantatively within the CFT. Hence we have come long way since Wheeler was quoted as saying "the question is - what is the question?" [26].

1.3 The minimalist's approach to holography

The last section paints a motivation for the connection between thermodynamics and black hole physics and a reason as to why it is particularly pertinent to black holes as potential gateways towards a better understanding of quantum gravity. The main question of this thesis is in a sense the reverse. The holographic principle of 't Hooft and Susskind suggests that the existence of a field theory dual is a generic feature of a gravitational theory. This raises a general question on conformal field theory:

> **main question:** How are the geometry and gravitational interactions of the bulk encoded in the universal features of candidate CFTs?

The process of black hole collapse is universal among all conventional gravitational theories of gravity, meaning those that at low energy scales reduce to Einstein gravity. The dual interpretation of gravitational collapse corresponds to the process of thermalization. The process whereby at the microscopic level an initial atypical state evolves and eventually languishes in a large class of typical states of a thermal ensemble. From the perspective of a macroscopic outside observer this would simply look like an initially disturbed active system which with time settles down into a calm thermal equilibrium.

The focus in particular will be one emergence of three-dimensional AdS_3 gravity from 2d conformal field theory. As is well known the algebra generating infinitessimal conformal transformations in two dimensions enhances to the infinite dimensional Virasoro algebra. This severe enhancement of the local symmetry algebra results in a large enhancement of the conformal representations. In particular this results in the fact that, unlike in higher dimension, under the state-operator map the stress tensor does not map to a highest weight state. In two dimensions the stress tensor is merely a descendant state of the vacuum. Not only that but all products and derivatives of the stress tensor are also contained within the highest weight representation generated from the vacuum. As was argued earlier the gravitational dual of the stress tensor is the graviton, hence the exchange of this one highest weight representation contains the full multi-graviton exchange of a scattering process in the bulk. This makes the AdS_3/CFT_2 correspondence an extremely efficient testing ground for studying gravity!

Importantly the results reported in this thesis will not depend on any particular CFT action, only general properties of certain CFTs will be assumed. Still there is a short list of assumptions that will be maintained on the class of CFTs under consideration.

- The theory is unitary
- The theory is conformal
- The central charge c will be very large
- There exists a gap in the operator spectrum
- A certain exponential scaling of conformal blocks is assumed (Zamolodchikov conjecture)

It was attempted to keep the list of assumptions as short as possible in order to fiducially claim that the obtained results are universal.

1.3.1 Outline

In an effort to keep this thesis largely self-contained it will start of with a few review chapters. Chapter 2 contains a review of 2d conformal field theory, with an emphasis on the Virasoro Ward identity and Virasoro coadjoint orbits. Chapter 3 covers some basics of asymptotically AdS_3 gravity including a modern review of the reappearance of the Virasoro algebra as the boundary charge algebra. The next chapter covers a review of semi-classical 2d

CFT, 2d conformal field theory where the central charge c is taken to be very large. In addition this chapter contains some ideas connecting uniformization to gravity. This chapter also contains a simple but novel derivation demonstrating the connection between uniformization, thermodynamics and 3d gravity.

After these review chapters the fifth chapter, based on the work [1], will give a description of thermalization in semi-classical CFT as a Markov chain process involving a certain monodromy matrix. It will also contain a Chern-Simons theory-based argument linking the late-time equilibrium temperature to the Hawking temperature of a bulk black hole associated to the Euclidean CFT stress tensor expectation value.

The sixth chapter is based on the work presented in [2]. It shows that generically CFTs contain a phase transformation from an ergodic to a thermalizing phase that has a particularly intuitive bulk interpretation. Given a scattering process in the bulk, it can potentially fail to form a black hole. If the same scattering experiment was to be repeated on AdS space containing a point mass at the origin then the result can be a black hole as long as the point mass exceeded a certain critical value.

Chapter 2

Introduction to Conformal field theory

2.1 The history of conformal field theory

Since the onset of quantum field theory the main guiding principle has been their underlying symmetries. To the point that it has been argued that quantum field theory is the only possible consistent construction combining locality, quantum mechanics and Lorentz invariance [3]. As a result it makes sense even from a purely academic point of view to consider if further increasing the level of symmetry leads to even more tightly constrained dynamical systems. One such non-trivial extension of the Lorentz group is the conformal group, which is obtained by relaxing the constraint that scalar products of vectors are preserved to merely the demand that normalized scalar products of vectors are preserved. A more intuitive way of phrasing the same statement is that Lorentz transformation preserve the lengths of vectors while the conformal group preserves angles.

The interest in these angle-preserving transformations has more applications than merely the academic though. Conformal field theory has a long history of physical interpretations and applications. The first recorded observation of the conformal group in physics was the observation that Maxwell's equations of electromagnetism are invariant under conformal transformations. Originally field theories invariant under conformal transformations were treated largely as a mathematical curiosity, providing quantum field theories that could be solved analytically[34]. The first point point where conformal invariance took off as subject with physical application in mind was in the study of second order phase transitions at the critical point. It was found in the Landau mean field description of condensed matter systems undergoing second order phase transitions that correlation lengths of fluctuations start to diverge near the critical point. This suggests that at the critical point the field theory becomes invariant under rescaling. While proven for two dimensions by Polchinski [36], it is generally also believed that in higher dimensions scale invariance along with the usual isometries of the field theory enhances to invariance under the full conformal group (barring some possible counterexamples [38]). To underlie the practical benefit of studying second order phase transitions through conformal field theory one can point out that the most accurate theoretical model for the critical exponents is due to conformal bootstrap, an associativity constraint on the operator algebra developed by Polyakov[39][40, 41].

The next major development in theoretical physics that would put conformal field theory on the map took place on the other side of the iron curtain. By the early 1970s the quantum theory of fields was in decline. Landau motivated by the vacuum polarization problem of QED[1], dismissed quantum field theory as a possible fundamental theory. Gell-mann proposed that the Lagrangian formulation of QFT should be thought of purely as a tool to get to the Feynman rules and ultimately the S-matrix [43]. This mindset changed with the advent of asymptotic freedom, which showed that even in the UV theories can flow to fixed points. Renormalization group flow paints a picture of a vast network where the lines are quantum field theories connected through the RG flow and the nodes are fixed points where the RG flow terminates. These nodes are the conformal field theories.

Of course in particular to the case of two-dimensional CFT a large impetus was imparted onto it's development by the motivation of string theory. While it was already noted after introduction of the Polyakov action that the quantum description of the string world sheet model induces a CFT, it was until the first string revolution that interest soared. The idea that the ultimate fundamental theory of nature is possibly given by a 2d CFT naturally pushed these theories to the foreground. In due time the first string revolution gave away to the second string revolutions, instigated by the discovery of D-branes. As was

[1] If a charge is placed in an otherwise vacuum, pair production polarizes the vacuum to shield the charge, if vacuum fluctuations of arbitrarily short wavelengths are taken into account ultimately the charge is screened al the way to zero. At the time this was incorrectly thought to be a universal feature of QFT rather than a quirk of Abelian gauge theory.

mentioned in the first chapter the discovery of D-branes eventually led to the discovery of AdS/CFT in which conformal field theory once again makes a key appearance. It is the AdS/CFT correspondence that provides the main motivation for the study conformal field theory in this thesis.

This chapter will review some of the basics of conformal field theory in two dimensions. The focus will be on the Virasoro Ward identity and its implications. As is somewhat unusual for a CFT review, a large amount of attention is devoted to the classification of the coadjoint orbits of the Virasoro algebra.

2.2 The conformal transformations in d dimensions

As mentioned in the introduction the set of conformal transformations is formed by the set of angle-preserving transformations. As a clarifying warm-up this short section will review the derivation of conformal transformation in more than two dimensions. This section is almost entirely based on the presentation in [45]. First recall the definition of the Lorentz transformations, these are all the coordinate transformations $x \to y(x)$ that leave the metric of Minkowski spacetime invariant. The metric, being a (2,0)-tensor of the spacetime manifold, transforms in the following way

$$\eta_{\mu\nu} \to \frac{dx^\alpha}{dy^\mu}\frac{dx^\beta}{dy^\nu}\eta_{\alpha\beta}. \tag{2.1}$$

Focussing on the infinitesimal transformations of the form $y^\mu = x^\mu + \epsilon^\mu(x)$ and substituting this back into the metric transformation above and demanding that the metric remains invariant gives

$$\begin{aligned}\eta_{\mu\nu} &= \left(\delta^\alpha_\mu - \partial_\mu\epsilon^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu\epsilon^\beta\right)\eta_{\alpha\beta} && (2.2)\\ &= \eta_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu + \mathcal{O}(\epsilon^2). && (2.3)\end{aligned}$$

Giving us the familiar result that the Lorentz transformations are infinitesimally generated by the $\frac{d}{2}(d-1)$ dimensional space of anti-symmetric $d\times d$ matrices. Determining the generators of the conformal transformations will follow an entirely analogous strategy. For that purpose consider the definition of an angle, consider two tangent vector fields $p^\mu(x)$ and $q^\mu(x)$ on some Lorentzian manifold. We can define the angle at the point x between non-lightlike vectors in the following geometric way

$$\cos(\theta(x)) = \frac{\langle p(x), q(x)\rangle}{|p(x)||q(x)|} = \frac{\eta_{\mu\nu}p^\mu(x)q^\nu(x)}{\sqrt{\eta_{\alpha\beta}p^\alpha p^\beta \eta_{\rho\sigma}q^\rho q^\sigma}}, \tag{2.4}$$

the relevant observation is that the right-hand side remains invariant if the metric field transforms up to a local Weyl rescaling

$$\eta_{\mu\nu} \to \Omega(x)\eta_{\mu\nu}. \tag{2.5}$$

Bringing the discussion back to Minkowksi space, this implies that the set of conformal transformations is comprised of the set of coordinate transformation that leave the metric invariant up to a local Weyl rescaling. One obvious intermediate conclusion is that the set of conformal transformations contains the set of Lorentz transformations. To contruct the infinitesimal conformal transformations we require that the metric transforms up to an infinitesimal Weyl rescaling $\eta_{\mu\nu} \to \eta_{\mu\nu} + \omega(x)\eta_{\mu\nu}$. Demanding that the coordinate transform $y^\mu = x^\mu + \epsilon^\mu(x)$ leads to a metric transformation that matches the form of the infinitesimal Weyl rescaling yields

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = -\omega(x)\eta_{\mu\nu}. \tag{2.6}$$

Taking the trace of this relationship above gives

$$\omega(x) = -\frac{2}{d}\partial^\mu \epsilon_\mu, \tag{2.7}$$

this lets us eliminate the Weyl factor from the equation (2.6)

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}\partial^\alpha \epsilon_\alpha \eta_{\mu\nu}. \tag{2.8}$$

Contracting the expression (2.8) with $\partial^\mu \partial^\nu$ gives us

$$\partial^2 \partial^\alpha \epsilon_\alpha = 0, \tag{2.9}$$

while contracting (2.8) with $\partial_\rho \partial^\mu$ leads to

$$\partial^2 \partial_\rho \epsilon_\nu + \left(1 - \frac{2}{d}\right) \partial_\rho \partial_\nu \partial^\alpha \epsilon_\alpha = 0, \tag{2.10}$$

symmetrizing this last expression in the indices and then combining these last two expressions brings us to the important conclusion that

$$\left(1 - \frac{2}{d}\right) \partial_\rho \partial_\nu \partial^\alpha \epsilon_\alpha = 0. \tag{2.11}$$

This implies that for $d > 2$ a conformal transformation can at most be quadratic in the old coordinates

$$\epsilon^\mu(x) = \alpha^\mu + \beta^\mu_\nu x^\nu + \gamma^\mu_{\nu\rho} x^\nu x^\rho, \tag{2.12}$$

substituting this expression back into (2.8) and solving for the coefficients α, β and γ gives the following classes of infinitesimal transformations

$$\begin{array}{lll} \text{Translations} & P_\mu & x^\mu \to x^\mu + \alpha^\mu, \\ \text{Lorentz tranformations} & M_\mu^\nu & x^\mu \to x^\mu + \omega_\nu^\mu x^\nu, \\ \text{Scale transformations} & D & x^\mu \to (1+\sigma)x^\mu, \\ \text{Special transformations} & K_\mu & x^\mu \to x^\mu + b^\mu x^2 - 2x^\mu b_\nu x^\nu. \end{array}$$

Exponentiating these infinitesimal transformation to full conformal transformations is mostly straightforward, with the exception of the special conformal transformations, their exponentiated conformal transformations are given by

$$\begin{array}{ll} \text{Translations} & x^\mu \to x^\mu + \alpha^\mu, \\ \text{Lorentz tranformations} & x^\mu \to M_\nu^\mu x^\nu, \\ \text{Scale transformations} & x^\mu \to \lambda x^\mu, \\ \text{Special transformations} & x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + x^\mu b_\mu + b^2 x^2}. \end{array}$$

We can obtain the coordinate representation of the generators by inserting their effect on the spacetime arguments of generic fields and Taylor expanding, e.g. scale transformations

$$f((1+\epsilon)x^\mu) = f(x^\mu) + \epsilon x^\mu \partial_\mu f(x^\nu) + \mathcal{O}(\epsilon^2), \tag{2.13}$$

from which we can read off that

$$D = -ix_\mu \partial^\mu, \tag{2.14}$$

the factor of i has been included to ensure the generator D a hermitian operator. By a similar calculation the full list of the coordinate representations of the generators is given by

$$\begin{array}{l} P_\mu = -i\partial_\mu \\ M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \\ D = -ix_\mu \partial^\mu \\ K_\mu = i(x^2 \partial_\mu - 2x_\mu x_\nu \partial^\mu) \end{array}$$

From these coordinate representations one can derive the following commutator algebra

$$\begin{array}{ll} [D, K_\mu] = -iK_\mu & [K_\mu, M_{\nu\rho}] = i\left(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu\right) \\ [D, P_\mu] = iP_\mu & [P_\rho, M_{\mu\nu}] = i\left(\eta_{\mu\nu}P_\nu - \eta_{\rho\nu}P_\mu\right) \\ [K_\mu, P_\nu] = 2i\eta_{\mu\nu}D - 2iM_{\mu\nu} & [M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}) \end{array}$$

where all commutators that are not listed are all equal to zero. This set of commutators forms the conformal algebra in $d > 2$. In Lorentzian signature the group generated from this algebra is isomorphic to $SO(2,d)$, in Euclidean signature it is isomorphic to $SO(1,d+1)$

2.3 Conformal transformations in two dimensions

The entirety of rest of the discussion in this chapter will be on conformal invariant field theories in two dimensions. More specifically the theories under consideration will be intended to live on the Euclidean cylinder parametrized by (τ,ϕ). Long before the AdS/CFT correspondence CFTs were studied on the cylinder in order to prevent spatial infra-red divergences [47], but for the purposes presented here it happens to make it particularly easy to identify the CFT spacetime with the spatial boundary of AdS$_3$ after analytic continuation to Lorentzian signature. As the cylinder is equivalent to the two-dimensional plane up to a global identification in the angular coordinate $\phi \sim \phi + 2\pi$ the cylinder just inherits the metric of the plane.

As the factor $(1 - 2/d)$ in equation (2.11) suggests, the set of conformal transformations is very different in two dimensions. Recalling the transformation of the metric field under the transformation $(\tau,\phi) \to (w_0(\tau,\phi), w_1(\tau,\phi))$ is given by (2.1), demanding that $g_{\mu\nu}(w) \propto g_{\mu\nu}(x)$ leads to the constraints

$$\frac{\partial w_1}{\partial \tau} = \pm\frac{\partial w_0}{\partial \phi}, \qquad \frac{\partial w_0}{\partial \tau} = \mp\frac{\partial w_1}{\partial \phi}. \tag{2.15}$$

These can be recognized as the Cauchy-Riemann relations, suggesting that after defining the complex coordinates

$$z = \tau + i\phi, \qquad \bar{z} = \tau - i\phi, \tag{2.16}$$

the set of all local conformal transformations defined around some open set containing a point z_0 consists of all (anti-)meromorphic functions of the z or $\bar{z}$ coordinates[2]. The emphasis on some open set around z_0 strikes as a rather cumbersome technical specification, but it is a very important one. The set of meromorphic functions on the entire complex plane fails to form a group, an easy way to see this is by noting that even the analytic functions of the

[2]We will analytically continue the coordinates z and $\bar{z}$ to full independent complex coordinates, 2d CFTs factorize into a left sector that lives on the z-plane and a right sector on the $\bar{z}$-plane, at the end of the day one has to reduce back to the 1-complex dimensional submanifold given by the condition $z^* = \bar{z}$.

complex plane fail to be invertible, the inverse of a single-valued mapping $w(z)$, i.e. $z(w)$ is not necessarily single-valued (e.g. $w(z) = z^2$ leads to $z(w) = \sqrt{w}$) and hence instead of mapping $\mathbb{C} \to \mathbb{C}$ it maps the complex plane to some multi-sheeted cover of the complex plane.

It turns out that the maximal subset of meromorphic functions that does form a group is given by the fractional linear transformations

$$w(z) = \frac{az+b}{cz+d}, \qquad ad - bc = 1. \tag{2.17}$$

The group formed by these functions is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2 \cong SO(1,3)$. As mentioned in previous section in d dimension the conformal algebra is $SO(1,d+1)$, this suggests that we can think of the algebra generated by the fractional linear transformations as the extrapolation of the algebra from the previous section to $d = 2$. It is the fractional linear transformations that form the true spacetime symmetries of the theory, as will be seen in a later section, these are the only conformal transformations that leave the vacuum invariant.

The fractional linear transformations contain poles, therefore the true set of spacetime symmetries possess the power to map a point to infinity and vice versa to map infinity to a finite location. This implies that CFTs naturally live not just on the (doubled) complex plane but on the (doubled) Riemann sphere, i.e. the complex plane with the point at infinity added.

To briefly bring the discussion back to fields, we can derive some intuition for the transformation of fields in a CFT from the d-dimensional algebra of the previous section. In that algebra the dilation generator D commuted with the full set of Lorentz transformation generators, as a result Schur's lemma implies that as a matrix acting on an irreducible representation of the Lorentz algebra D has to take the form of a multiple of the identity matrix $D = -i\Delta I$. This suggests the intuitive fact that to every Lorentz multiplet in our action we attribute a number, the scaling dimension, that tells us how the corresponding field transforms under rescalings

$$\phi(\lambda x) = \lambda^{-\Delta}\phi(x). \tag{2.18}$$

The action of the special conformal transformation generators suggests that there exists a subset of fields in the spectrum where the transformation rule above is extrapolated to all the conformal transformations

$$\phi(w(x)) = \det\left(\frac{dx^\mu}{dw^\nu}\right)^{-\Delta/d}\phi(x), \tag{2.19}$$

these form the so-called primary fields of the spectrum. In two dimension this transformation law gets extended to include the spin of the fields, define the (anti-)holomorphic scaling dimensions through the linear combinations

$$h = \frac{1}{2}(\Delta + s) \qquad \bar{h} = \frac{1}{2}(\Delta - s). \tag{2.20}$$

In which case (quasi-)primary fields are those that transform as

$$\phi(w(z), \bar{w}(\bar{z})) = \left(\frac{dz}{dw}\right)^{-h} \left(\frac{d\bar{z}}{d\bar{w}}\right)^{-\bar{h}} \phi(z, \bar{z}), \tag{2.21}$$

if the transformation rule above holds for all conformal transformations the field ϕ is dubbed a primary field, if it merely holds for all the fractional linear transformations it is dubbed a quasi-primary field. The role of the distinction will be expanded upon in section 2.6 where Virasoro representation theory will be discussed. Taylor expanding the above expression gives the effect of an infinitesimal conformal transformation $\epsilon, \bar{\epsilon}$ on a primary field

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = -h(\partial_z \epsilon)\phi(z,\bar{z}) - \bar{h}(\partial_{\bar{z}}\bar{\epsilon})\phi(z,\bar{z}) - \epsilon\partial_z\phi(z,\bar{z}) - \bar{\epsilon}\partial_{\bar{z}}\phi(z,\bar{z}), \tag{2.22}$$

this expression will play an important role in identifying the conformal Ward identity.

2.4 The 2d conformal Ward identity

As shown in the last section, the set of conformal transformations in two dimension is very wide. The next logical step is to determine what kind of constraints these symmetries impart on observables in a quantum theory. In general the effect of a symmetry transformation on a correlation function is given by the Ward identity

$$\partial_\mu \langle J^\mu \phi_1 ... \phi_n \rangle = -i \sum_{k=1}^{n} \delta(x - x_k) \langle \phi_1 ... G\phi_k ... \phi_n \rangle, \tag{2.23}$$

where J^μ is the conserved Noether current of the symmetry and G is the generator of the symmetry acting on the field operators. Unfortunately deriving the conformal Noether current of a generic conformal transformation from an action containing generic fields is not such a useful strategy. The reason for this is due to the fact in general the fields upon which the action depends are not the (quasi-)primary fields of the theory. This makes it hard to tell how these fields will transform under general analytic mappings. Examples of conformal field theories whose action functional fields are not the (quasi-)primary fields includes for instance the free boson and Liouville field theory.

To circumvent this problem we can make use of the fact that locally an analytic function is nothing but a combination of a translation, rotation and a rescaling. This can be seen by expanding an analytic function $f(z)$ around a point z_0

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \mathcal{O}\left((z - z_0)^2\right), \tag{2.24}$$

hence an infinitesimal shift vector attached to z_0 in the transformed coordinates is related to the shift vector in the old coordinates by multiplication with the complex number $f'(z_0)$, multiplying a complex number by another complex number has the effect of rotating and rescaling that number. Therefore the full local effect of a conformal transformation is to translate the point z_0 to $f(z_0)$ and to rotate and rescale the tangent space at z_0. Fortunately, general fields do transform in predictable ways under both rotations and scale transformations. Both related Ward identities will be briefly covered first

Rotational invariance Ward identity

The current under rotations in d dimensions is given by the familiar expression

$$J^{\mu}_{\rho\sigma} = x_\rho T^{\mu}_{\sigma} - x_\sigma T^{\mu}_{\rho} + \frac{i}{2}\sum_{\phi}\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}S_{\rho\nu}\phi, \tag{2.25}$$

where the sum runs over all fields that appear within the Lagrangian. Inserting this current into the Ward identity above leads to the following general rotation Ward identity

$$\langle (T_{\mu\nu} - T_{\nu\mu})\phi_1 ... \phi_n\rangle = -\frac{i}{2}\sum_{k=1}^{n}\delta(x - x_k)\langle\phi_1 ... S^{k}_{\mu\nu}\phi_k ... \phi_n\rangle \tag{2.26}$$

expressing the familiar fact that the stress-tensor expectation value is symmetric up to contact terms. This expression can be simplified in two dimensions, where the rotation group is only one-dimensional, the spin connection is fixed to the simple form $S_{\mu\nu} = s\varepsilon_{\mu\nu}$, where s is the spin of the field. By making use of the identity $\varepsilon_{\mu\alpha}\varepsilon^{\alpha\nu} = -\delta^{\nu}_{\mu}$ the above expression simplifies to

$$\varepsilon_{\mu\nu}\langle T^{\mu\nu}\phi_1 ... \phi_n\rangle = -i\sum_{k=1}^{n} s_k\delta(x - x_k)\langle\phi_1 ... \phi_n\rangle. \tag{2.27}$$

Scale invariance Ward identity

One can similarly derive the Ward identity associated to scale transformations. The Noether current resulting from the scale invariance of an action is given by

$$j_D^\mu = T^\mu_\nu x^\nu + \sum_\phi \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \Delta\phi, \tag{2.28}$$

this is not the form of the current generally used though, analysing the effect of demanding scale invariance of the Schwinger function $\langle T_{\mu\nu}(x) T_{\rho\sigma}(y)\rangle$ shows that in 2d the vacuum expectation value of the trace squared $(T^\mu_\mu)^2$ has to vanish. While this is in no means constitutes a prove it is suggestive towards a general statement that scale invariance implies that the stress tensor can be made traceless. By combining the fact that the variation of an action under diffeomorphism is proportional to stress tensor with the definition of an infinitesimal conformal transformation (2.6) one can show that tracelessness of the stress tensor implies conformal invariance. This assumption that scale invariance implies that the stress tensor can be made traceless would therefore ensure that scale invariance implies conformal invariance, but that is alright since this was proven to be true in two dimensions through alternative means [36, 37]

As such one generally considers the simpler scale transformation current

$$j_D^\mu = T^\mu_\nu x^\nu, \tag{2.29}$$

in which case inserting this current into the Ward identity (2.23) leads to

$$\langle T^\mu_\mu \phi_1 ... \phi_n\rangle = -\sum_{k=1}^n \delta(x - x_k)\Delta_k \langle \phi_1 ... \phi_n\rangle. \tag{2.30}$$

Ward identity of general conformal transformations

The results of the last two sections can be combined to derive the Ward identity associated to general conformal transformations. Take the general infinitesimal conformal transformation $\epsilon_\mu(x)$, take the expression $\partial_\mu \epsilon_\nu$ and decompose it into its symmetric and anti-symmetric part

$$\partial_\mu \epsilon_\nu = \frac{1}{2}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \frac{1}{2}(\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu), \tag{2.31}$$

the first term on the right-hand side can be reduced through (2.8). The second term requires the identity

$$\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu = \varepsilon^{\alpha\beta} \partial_\alpha \epsilon_\beta \varepsilon_{\mu\nu}, \tag{2.32}$$

which can be shown to hold true in two dimensions by substituting the definition of the anti-symmetric tensor into the right-hand side. After substituting (2.8) and (2.32) and multiplying with $T^{\mu\nu}$ equation (2.31) takes the form

$$T^{\mu\nu}\partial_\mu\epsilon_\nu = \frac{1}{2}(\partial_\alpha\epsilon^\alpha)g_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\varepsilon_{\mu\nu}T^{\mu\nu}, \tag{2.33}$$

this expression is reminiscent of the earlier statement that a generic analytic transformation locally resembles a scaled rotation. Inserting both sides of the above expression into a generic correlation function and integrating over an arbitrary region of the spacetime cylinder that contains all the fields and applying the Ward identities (2.27) and (2.30) gives

$$\begin{gathered}\int d^2x\ (\partial_\mu\epsilon_\nu)(x)\langle T^{\mu\nu}(x)\phi_1...\phi_2\rangle = \\ -\tfrac{1}{2}\sum_{k=1}^n\left((\partial_\alpha\epsilon^\alpha)|_{x_k}\Delta_k + i\varepsilon^{\alpha\beta}(\partial_\alpha\epsilon_\beta)|_{x_k}s_k\right)\langle\phi_1...\phi_n\rangle.\end{gathered} \tag{2.34}$$

If we allow the derivative on the left-hand side to act on the full left-hand side as opposed to the conformal transformation and apply the translation Ward identity we obtain

$$\begin{gathered}\int d^2x\,\partial_\mu\epsilon_\nu(x)\langle T^{\mu\nu}(x)\phi_1...\phi_2\rangle = \\ -\sum_{k=1}^n\left(\epsilon^\nu\frac{\partial}{\partial x_k^\nu} + \tfrac{1}{2}(\partial_\alpha\epsilon^\alpha)|_{x_k}\Delta_k + \tfrac{i}{2}\varepsilon^{\alpha\beta}(\partial_\alpha\epsilon_\beta)|_{x_k}s_k\right)\langle\phi_1...\phi_n\rangle.\end{gathered} \tag{2.35}$$

This entire expression is of the form of an integrated Ward identity, since the effect a conformal transformation can have on the fields is exhausted by the combination of a translation and a scaled rotation we interpret this expression as the Ward identity associated to a conformal transformation infinitesimally generated by $\epsilon_\mu(x)$. This suggests that the conformal current is of the form

$$j^\mu_\epsilon = \epsilon_\nu(x)T^{\mu\nu} \tag{2.36}$$

At this point it is necessary to stress a small point: we did not derive the conformal current, instead we noted that locally a conformal transformation resembles a translation followed by a scaled rotation and we noticed that inserting the expression (2.36) into a generic correlator has the effect of resembling a Ward identity whose field transformation has the effect of a translation combined with a scaled rotation. This leads us to postulate that the conformal current is given by (2.36). The complete justification of this claim can be obtained by going to the complex plane.

2.4.1 Conformal Ward identity on the complex plane

The ward identity reviewed so far was given in terms of two arbitrary real coordinates, but as discussed in section 2.3 a 2d CFT naturally lives on the complex Riemann sphere with (anti-)holomorphic complex coordinates z, $\bar{z}$ defined through (2.16). In this coordinate system the metric components are given $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$, with the other components equal to zero. Demanding that stress tensor is traceless in symmetric leads to some constraints on the components of the stress tensor, namely $T_{z\bar{z}} = T_{\bar{z}z} = 0$, demanding that the conformal current (2.36) is conserved leads to the constraints [47]

$$\partial_{\bar{z}} T_{zz} = 0, \qquad \partial_z T_{\bar{z}\bar{z}} = 0. \tag{2.37}$$

These two things combined state that on-shell the stress tensor only has two independent degrees of freedom and each of these degrees of freedom are respectively either holomorphic or anti-holomorphic

$$T(z) \equiv -2\pi T_{zz}, \qquad \bar{T}(\bar{z}) \equiv -2\pi T_{\bar{z}\bar{z}}. \tag{2.38}$$

These objects will be of great importance in the later chapters, in the semi-classical CFTs discussed in chapters 4,5 and 6 these functions can be interpreted as the fundamental degrees of freedom with a phase space given by the set of Virasoro coadjoint orbits.

As the form of the left-hand side of the Ward identity (2.35) suggests, it can be reduced to a surface integral by means of Gauss' theorem. On the complex plane it can be reduced to a sum of contour integrals

$$\begin{aligned} -\tfrac{1}{2\pi i} \oint dz\, \epsilon(z) \langle T(z)\phi_1 ... \phi_n\rangle + \tfrac{1}{2\pi i} \oint d\bar{z}\, \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z})\phi_1 ... \phi_n\rangle = \\ -\textstyle\sum_{k=1}^{n} \left(\epsilon \frac{\partial}{\partial_{z_k}} + \bar{\epsilon} \frac{\partial}{\partial_{\bar{z}_k}} + h \left(\partial_z \epsilon(z)\right)|_{z_k} + \bar{h} \left(\partial_{\bar{z}} \bar{\epsilon}(\bar{z})\right)|_{\bar{z}_k} \right) \langle \phi_1 ... \phi_n \rangle, \end{aligned} \tag{2.39}$$

where the scaling dimensions of (2.20) have been reinstated. This is the conformal ward identity on the complex plane for correlators containing primary fields, it is an indispensable tool in the sense that almost everything else mentioned in the later chapters ultimately stems from this formula. For one, the right hand side can be identified to be equal to the infinitesimal transformation rule (2.22) for a product of primary fields, reinforcing the notion that the conformal current is given by $J(z) = \epsilon(z)T(z)$.

As a straightforward consequence one can find an expression for the effect of inserting the stress tensor into a correlator that consists entirely of primary fields. To accomplish this specify $\epsilon(z) = 1/(w-z)$ and $\bar{\epsilon} = 0$ in (2.39) to obtain

$$\langle T(w)\phi_1...\phi_n\rangle = \sum_{k=1}^{n}\left(\frac{h_i}{(w-z_k)^2} + \frac{\partial_{z_k}}{w-z_k}\right)\langle\phi_1...\phi_n\rangle + \text{reg.} \tag{2.40}$$

The reg. term indicates regular terms, the contour integral is insensitive to these terms which as a result leaves these terms unspecified. They can be constructed by means of the the Virasoro algebra though, this algebra will be the subject of the next section.

2.5 The Virasoro algebra

The set of conformal transformations considered so far is somewhat unusual for a spacetime symmetry. They are generated by an infinite dimensional linear space and most of them cannot be exponentiated to group elements on the complex plane. But we do know a general form for the conformal current which allows us to study what happens when we conjoin two infinitesimal conformal transformations. First to exploit the power of contour integrals on the complex plane perform a conformal transformation to the radial plane through the transformation

$$z = e^{\tau+i\phi}, \quad \bar{z} = e^{\tau-i\phi}. \tag{2.41}$$

This coordinate system rescales every time slice to a circle of different size and the resulting foliation in the time direction is given by the radial coordinate. This coordinate system has some extremely important implications such as the state-operator correspondence, but the most immediately useful implication is related to the conformal charges. In general given a conserved current we can construct an associated conserved charge

$$Q_a(x) = \int d^{d-1}x j_a^0(x), \tag{2.42}$$

where the integral runs over some fixed time slice. The integral is particularly convenient on the radial plane since an integral over a fixed time slice takes the form of an integral along a circle with a radius corresponding to time. If the spatial slice is taken to be the unit circle then in the radial z, $\bar{z}$ coordinates the component of the current normal to the circle is

$$j_\epsilon^0(z,\bar{z}) = z\epsilon(z)T^{zz} + z\bar{\epsilon}(\bar{z})T^{z\bar{z}} + \bar{z}\epsilon(z)T^{\bar{z}z} + \bar{z}\bar{\epsilon}(\bar{z})T^{\bar{z}\bar{z}}, \tag{2.43}$$

demanding stress tensor to symmetric and traceless corresponds to $T^{z\bar{z}} = T^{\bar{z}z} = 0$, this combined with the definitions (2.38) allows the conformal charges to be written in terms of the following contour integrals

$$Q_\epsilon[T] = \frac{1}{2\pi i} \oint dz\, \epsilon(z) T(z). \tag{2.44}$$

This is the power of the fact that 2d CFT naturally lives on the complex Riemann sphere, the charge algebra inherits all of the structure and analytical tools of complex analysis. Integrating the Ward identities along a fixed time slice suggest that the charge operators are closely related to inftinitessimal symmetry generators through $[Q_a, \phi] = -iG_a\phi$. This provides us with a pathway to derive the symmetry algebra associated to the infinitesimal conformal transformations. The conformal charges posses a natural Lie bracket structure by considering the effect of a conformal transformation on the charge of another conformal transformation i.e.

$$[Q_{\epsilon_1}, Q_{\epsilon_2}][T] = -Q_{\epsilon_1}[\delta_{\epsilon_2} T]. \tag{2.45}$$

This suggests that constructing the conformal algebra corresponds to finding the transformation rules for the holomorphic component of the stress tensor. In fact it is simpler than that, the contour integral in (2.44) only picks out the residues of poles hence we only need the singular terms of the transformation rule. Note that equivalent considerations apply for the anti-holomorphic part.

2.5.1 Transformation properties of the stress tensor

Consider the product of operators $T(z)T(w)$, the stress tensor is not a primary in field in 2d CFT, as a result expression (2.40) cannot be applied. By the fact that the stress tensor represents a density and demanding scale invariance it can be found that singular terms of $T(z)T(w)$ are fixed to

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg.} \tag{2.46}$$

Here c is the ubiquitous central charge that appears throughout 2d conformal field theory. Note that if $c = 0$ the stress tensor would be a primary field with scaling dimension $h = 2$. From this product we can derive the infinitesimal transformation rule for the stress tensor through means of the conformal charge (2.44)

$$\delta_\epsilon T(z) = [Q_\epsilon, T(z)] = \frac{1}{2\pi i}\oint dw\, \epsilon(w) T(w) T(z)$$
$$\frac{1}{2\pi i}\oint dw\, \epsilon(w)\left(\frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial_z T(z)}{z-w}\right). \tag{2.47}$$

At this point simply collecting the residues leads to the following infinitesimal transformation rule

$$\delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2T(z)\partial\epsilon(z) + \frac{c}{12}\partial^3\epsilon(z). \tag{2.48}$$

It is this expression that will let us identify the set of conformally related stress tensor expectation values to the Virasoro coadjoint orbits in section 2.8. Exponentiating this infinitesimal transformation rule to a transformation under large meromorphic transformation is a difficult problem, but one whose solution is famously presented in the seminal work of Belavin, Polyakov and Zamolodchikov [33]. Under general conformal transformation the stress tensor transforms as

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z(w)) + \frac{c}{12}S[z,w], \tag{2.49}$$

where $S[z,w]$ is the Schwarzian derivative given by

$$S[z,w] = \frac{z'''(w)}{z'(w)} - \frac{3}{2}\left(\frac{z''(w)}{z'(w)}\right)^2, \tag{2.50}$$

the Schwarzian derivative satisfies a particular chain rule property

$$S[u(w),z] = S[w,z] + \left(\frac{dw}{dz}\right)^2 S[u,w], \tag{2.51}$$

this property ensures that the transformation rule of two successive conformal transformation is given by that of the conjoined conformal transformation. Schwarzian derivative satisfies another property, it vanishes identically for fractional linear transformations, meaning that the stress tensor falls in the category of quasi-primary operators.

2.5.2 Virasoro algebra

The bracket structure (2.45) along with the transformation rule for the stress tensor under infinitesimal conformal transformations (2.48) is enough to derive the 2d conformal algebra. The set of all meromorphic functions is generated by an infinite dimensional space, take as a basis for this space the set of Laurent modes of a meromorphic function, i.e. consider the following basis for set of infinitesimal transformations $\epsilon_n(z) = z^{n+1}$, where n can be any integer, positive or negative. This gives us the accompanying set of conformal charges

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z), \tag{2.52}$$

conveniently this is exactly the form of the inverse Laurent transformation of the stress tensor, suggesting that the L_n operators function as the Laurent coefficients of $T(z)$

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}. \tag{2.53}$$

The set of transformations $\epsilon_n(z) = z^{n+1}$ furthermore suggests that L_{-1} is momentum charge while L_0 plays the role of the dilation charge. Inserting the charges L_n into (2.45) and applying (2.48) leads to the following conformal algebra

$$\boxed{[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m},} \tag{2.54}$$

This is the famous Virasoro algebra of 2d conformal field theory. It possesses an equivalent anti-holomorphic counterpart

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}, \tag{2.55}$$

where $\bar{L}_n$ modes are constucted out of $\bar{T}(\bar{z})$ in a way entirely anologous to (2.52), furthermore

$$[L_n, \bar{L}_m] = 0. \tag{2.56}$$

Imposing the real surface condition $z^* = \bar{z}$ clarifies that the set of physical transformations is given by $L_n + \bar{L}_n$ and $i(L_n - \bar{L}_n)$, telling us that conformal algebra is given by a doubled Virasoro algebra. The generators L_{-1}, L_1, L_0 form a closed subalgebra for which the central extension term vanishes, this subalgebra, called the global subalgebra, generates the fractional linear transformations (2.17).

This central extension term is atypical for spacetime symmetry algebras. That is because technically the Virasoro algebra is not the spacetime symmetry algebra of 2d CFT but rather the Dirac algebra of the conserved charges of the theory. Often the algebra of conserved charges is of more direct physical interest in a local quantum field theory as it provides direct information on the correlators of the charge operators. But while the charge algebra is very closely related to the spacetime algebra they are not exactly isomorphic. In fact represenations of the charge algebra only form projective representations of the spacetime algebra. It is this extra dimension in the charge representation space that results in a central extension term for the charge algebra [18]. The same issue already occurs in classical Hamiltonian mechanics, the Poisson bracket algebra of Hamiltonian functions on the phase space is related to the Lie algebra of tangent vectors by pushing the exterior derivative of the Hamiltonian function through the symplectic form. Adding a constant to the Hamiltonian function does not affect the Hamiltonian phase flow, this many-to-one property between Hamiltonian functions and Hamiltonian phase flows causes the representations of the Poisson bracket to be a projective representations of the Lie algebra representations. See chapter 40 of [49] for details.

2.6 Virasoro representation theory

The next logical step after constructing the conformal algebra is to construct the representations of this algebra, for this purpose a special role is played by the primary fields of section 2.3.

2.6.1 primary fields and their alternatives

In the last sections a special role was played by the primary fields $O(z)$, these were defined by the transformation rule[3]

$$O(w) = \left(\frac{dz}{dw}\right)^h O(z(w)), \tag{2.57}$$

which result in the following infinitesimal transformation rule

$$\delta_\epsilon O(z) = -h(\partial_z \epsilon) O(z) - \epsilon \partial_z O(z). \tag{2.58}$$

The set of primary fields are not the most general fields that can be contained with a CFT, the stress tensor $T(z)$ as was encountered before is not a primary field. To consider the alternatives to primary fields imagine the possible infinitesimal transformation rules of some generic field $\tilde{O}$ compliant with scale and translation invariance

$$\delta_\epsilon \tilde{O}(z) = -h(\partial_z \epsilon)\tilde{O}(z) - \epsilon(z)\partial_z \tilde{O}(z) - \sum_{n=2} c_n (\partial_z^n \epsilon(z)) f_n(\partial_z) \tilde{O}(z), \tag{2.59}$$

[3]Suppressing the anti-holomorphic sector

the first term on the right hand-side is fixed by rotation and scale invariance. After imposing the real surface condition the rescaling and rotation generators are respectively given by $L_0+\bar{L}_0$ and $i(L_0-\bar{L}_0)$ these clearly commute, therefore any fields that form a representation under rotations by Schur's lemma also possess a scaling dimension Δ and we can consistently associate numbers $h, \bar{h}$ to any physical field, whether primary or not. The remainder of the possible terms in the expression above are fixed by translation invariance or restricted by the demand that we only keep the linear terms in $\epsilon(z)$, the operators $f_n(\partial_z)$ form some generic polynomials of derivative operators. Repeating the steps that led to the TO operator product expansion leads to

$$T(z)\tilde{O}(w) = \left(\frac{\partial_w}{z-w} + \frac{h}{(z-w)^2} + \sum_{n=2} \frac{n! c_n f_n(\partial_w)}{(z-w)^{n+1}} \right) \tilde{O}(w) + \text{reg.} \tag{2.60}$$

This elucidates an alternative view of the primary fields, they are the fields whose TO operator product expansion, discounting the identity operator, contains the minimal amount of singular terms.

2.6.2 The Hamiltonian and raising and lowering operators

The name radial plane derives from the fact that time runs in the radial direction, hence it is the dilation generator L_0 that takes us from one time slice to the next, i.e. L_0 plays the role of the Hamiltonian. Assuming that the CFT has a unique vacuum state, L_0 generates an exact symmetry of the theory, therefore if we assume that the vacuum does not break conformal invariance then $L_0|0\rangle = 0$. Consider the state created by acting on the vacuum with a generic operator inserted at the origin $\tilde{O}(0)|0\rangle$, acting on this state with L_0 gives

$$L_0 O(0)|0\rangle = \frac{1}{2\pi i} \oint dz\, z T(z) O(0)|0\rangle = h O(0)|0\rangle, \tag{2.61}$$

here the second step can be obtained by applying (2.60). Hence field operators inserted at the origin form eigenstates of the Hamiltonian with eigenvalue h. States created by acting on the vacuum with an operator at the origin form the asymptotic states of the CFT, since they are energy eigenstates created at the infinite past in Euclidean time. Unlike in usual perturbative QFT terminology these asymptotic states are the exact eigenstates of the full interacting Hamiltonian as opposed to some free Hamiltonian [47].

The commutator of L_n with any other Virasoro generator L_n

$$[L_0, L_n] = -n L_n, \tag{2.62}$$

suggests that the other Virasoro operators can be interpreted as raising or lowering operators depending on the sign of n. In order to construct a unitary representation of the Virasoro algebra we have to identify the highest weight states, i.e. those states that are annihilated by all the lowering operators. The Virasoro generators acting on an energy eigenstate

$$L_n O(0)|0\rangle = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z) O(0)|0\rangle, \tag{2.63}$$

will annihilate the state for $n > 0$ if the TO expansion contains at most a double pole at the origin. Meaning that the set of field operators that create highest weight states when inserted at the origin is the set of primary operators. This can be interpreted as a third alternative definition for primary operators.

All energy eigenstates can be classified in either one of two categories, primary states, those created by primary operators at the origin $|h\rangle$ or descendent states, so called because they can be contructed out of primary states by acting with combinations of Virasoro raising operators on a primary state $L_{-n}...L_{-m}|h\rangle$. Acting with a raising mode L_{-n} on a primary state increases the scaling dimension of that state by n units, hence we can sort the descendent states by level. The set of descendent states at a given level form a linear space, without introducing an operator ordering the states at a given level will form an overcomplete basis due to the Virasoro algebra, consider as an example the following state at level 3

$$L_{-2}L_{-1}|h\rangle = L_{-1}L_{-2}|h\rangle - L_{-3}|h\rangle, \tag{2.64}$$

to avoid this issue conventionally the following ordering is maintained when constructing descendent states namely $L_{-n_1}L_{-n_2}...L_{-n_k}|h\rangle$ where $n_k \geq n_{k-1} \geq ... \geq n_1$. With the ordering imposed the number of states at any level n is given by number of ways the integer n can partioned into a sum over other integers (e.g. 3=1+1+1=1+2). Even after imposing this ordering it is still possible for the states at a given level to be linearly dependent, this implies that at a given level a linear combination of states can be formed that add up to a null state, in general the number of linearly independent states at a given level is encoded within the Taylor series coefficients of a function called character $\chi_h(q)$ of a conformal family. We will come back to the presence of these null states in chapter 4.

In thermal field theory the central object of study is the partition function $Z(\beta) =\mathrm{Tr}\big(e^{-\beta H}\big)$, which means that it is relevant to get a sense of scale for how fast the number of states grows within a conformal family (a primary state with all of its descendants). The asymptotic scaling of the number of states $P(n)$ at a given level n starts to approach the Hardy-Ramanujan formula [48]

$$P(n) = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} + \mathcal{O}(\sqrt{n}). \tag{2.65}$$

the partition function of a 2d CFT can be expressed as a sum over conformal families parametrized by the scaling dimension of the conformal primaries

$$Z(\beta) = Tr\left(e^{-\beta L_0}\right) = \sum_h e^{-\beta h} \sum_{n=0}^{\infty} e^{-\beta n} P(n) \tag{2.66}$$

In the high temperature regime $\beta \ll 1$ the sum over n can be approximated by saddle-point analysis. By inserting the Ramanujan-Hardy formula one can find that the sum over n is dominated roughly around $n = \frac{\pi^2}{6\beta^2}$, which gives

$$Z(\beta \ll 1) = \frac{\sqrt{3}\beta^2}{2\pi^2} \sum_h \exp\left(\frac{\pi^2}{6\beta} - \beta h\right). \tag{2.67}$$

Of particular interest will be those conformal field theories that potentially possess a gravitational dual. Despite the fact that the number of descendent states grows very rapidly as we increase the level, the partition function still woefully underestimates the high temperature partition function predicted either by the Cardy formula or, equivalently, the Bekenstein-Hawking entropy if we assume the CFT Hilbert space consists of only a small number of conformal families. As we shall get back to in chapter 3, to acount for the number of high temperature black hole states a 2d CFT needs to contain a very sharp growth of the number of conformal families as a function of h.

2.7 Conformal field theory dynamics

The last chapter focussed on the Hilbert space of a CFT, in this chapter it is time to instead discuss the dynamics of the field theory. As is typical in a quantum theory the full dynamics of the theory are encoded within its correlation functions, hence it is often said that a theory is fully solved whenever all possible correlators are known. While generally a hopeless task in any interacting quantum field theory, where finding any non-trivial exact correlator is a non-trivial achievement, conformal field theories stand as exceptions. CFTs stand alongside integrable models as examples of often strongly interacting theories where constructing the full set of correlators is not entirely futile. In fact in case of the minimal models, where the number of conformal families in the Hilbert space is restricted to a finite number, the theories have been solved entirely [33].

2.7.1 the 2- and 3-point functions and the adjoint operation

Any generic CFT provides examples of simple correlators that are entirely fixed. The transformation rule of quasi-primary operators (2.21) combined with conformal invariance under the fractional linear transformations allows one to completely fix the two-point functions of primary operators to

$$\langle O_{h_1}(z)O_{h_2}(w)\rangle = \frac{\delta_{h_1h_2}}{(z-w)^{2h_1}}, \tag{2.68}$$

and fixes the three-point functions up to an overall coefficient

$$\langle O_{h_1}(z)O_{h_2}(w)O_{h_3}(u)\rangle = \frac{c_{123}}{(z-w)^{h_1+h_2-h_3}(w-u)^{h_2+h_3-h_1}(u-z)^{h_1+h_3-h_2}}, \tag{2.69}$$

where as usual the equivalent anti-holomorphic factor has been suppressed. While fixing only the primary operator low point correlators might seem initially restrictive, in fact this fixes the two- and three-point functions of all operators due to the conformal Ward identity

So far all mention of the adjoint states has been avoided, the analytic form of the two-point function hints at what they should be though. A primary state prepared in the asymptotic past $|h\rangle = O_h(0)|0\rangle$ should possess an appropriate adjoint state $\langle h| = \langle 0|[O_h(0)]^\dagger$. The in-state corresponds to excitation created in the asymptotic past, hence the out-state should correspond to an excitation annihilated at the infinite future, on the radial plane time-reversal $\tau \to -\tau$ corresponds to $z \to \frac{1}{z^*}$. If we therefore define

$$[O_h(z)]^\dagger = (z^*)^{-2h}\, O_h\left(\frac{1}{z^*}\right), \tag{2.70}$$

the scalar product of two primary states takes the expected form

$$\langle h'|h\rangle = \lim_{z\to 0}\,(z^*)^{-2h}\,\frac{\delta_{h'h}}{(1/z^*-z)^{2h}} = \delta_{h'h}. \tag{2.71}$$

The form of (2.70) is misleadingly similar to that of the transformation rule of the primary field. Time-reversal mixes the holomorphic and anti-holomorphic coordinates and as such is not a conformal transformation, the similarity is due to the fact that the primary transformation rule fixes the two-point function and in return we have used the two-point function to fix the adjoint operation.

An example of a quasi-primary operator is the stress tensor $T(z)$, applying the operation (2.70) provides us with the important relationship

$$[T(z)]^\dagger = (z^*)^{-4} T\left(\frac{1}{z^*}\right), \tag{2.72}$$

inserting the Laurent series decomposition of the stress tensor (2.53) on both the left hand- and right hand-side constrains the Virasoro generators to satisfy

$$L_n^\dagger = L_{-n}. \tag{2.73}$$

The normalization (2.71) not only states that primary states are mutually orthogonal but that entire Virasoro representations are orthogonal. This can schematically be made visible by computing the scalar products of two generic descendant states $L_{-n}...L_{-m}|h\rangle$ and $L_{-k}...L_{-p}|h'\rangle$

$$\begin{gathered}\langle h'|L_p...L_k L_{-n}...L_m|h\rangle = \\ \prod_i \tfrac{1}{2\pi i} \oint dz_i\, z_i^{n_i+1} \left(\sum \tfrac{h}{(1/w^*-z_i)^2} + \tfrac{\partial_w}{w-z_i}\right) \langle h'|h\rangle \propto \delta_{h'h},\end{gathered} \tag{2.74}$$

any two states that do not belong to the same unitary Virasoro representation are orthogonal to one another.

2.7.2 state-operator correspondence

A crucial feature, unique to conformal field theory, is that there exists a bijective mapping from the states of the theory to the local operators of the theory. This feature ultimately follows from the presence of scale invariance. The basis of radial quantization was due to the fact that scale invariance implies that there exists a hermitian operator that pushes a state that lives on a circle to a state that lives on an infinitesimally larger circle. Conversely we can take limit where Euclidean time goes to the infinite past in which case the whole time slice contracts to a single point at the origin. Consider a generic state on a time slice parametrized by $z = e^{\tau + i\phi}$

$$|\psi(\tau')\rangle = \int_0^{2\pi} d\phi \sum_i g_i(\phi) O_i(e^{\tau' + i\phi})|0\rangle, \tag{2.75}$$

evolving this state to the asymptotic past by means of the Hamiltonian L_0

$$\begin{aligned}\lim_{\tau\to-\infty} e^{-\tau L_0}|\psi(\tau')\rangle &= \lim_{\tau\to-\infty} \int_0^{2\pi} d\phi \sum_i g_i(\phi) e^{-\tau L_0} O_i(e^{\tau'+i\phi}) e^{\tau L_0}|0\rangle \\ &= \lim_{\tau\to-\infty} \int_0^{2\pi} d\phi \sum_i g_i(\phi) O_i(e^{\tau} z)|0\rangle \equiv O'(0)|0\rangle, \end{aligned} \tag{2.76}$$

in the last step above the limit τ to negative infinity contracts the radius of the circle on which the operators are inserted to zero, hence we can define a regularized local operator $O'(0)$ that captures the full aggregated effect of the contracted operator. While it is generally possible to associate an operator to a state, what separates conformal field theory is that we can associate a *local* operator to a state.

One can repeat the analysis after replacing the state $|\psi\rangle$ by an L_0 eigenstate $|h\rangle$. These eigenstates are stationary states, i.e. $\exp(-\tau L_0)|h\rangle = \exp(-\tau h)|h\rangle$, hence after taking $\tau \ll 0$ where the state operator can be replaced by an aggregate local operator $O'(z)$ the only way we can maintain the same scaling with τ on the left and right hand-side is if $O'(e^\tau z) = \exp(-\tau h) O'(z)$, i.e. $O'(z)$ transforms under scale transformations as an operator with definite scaling dimension h. This conclusion implies that the set eigenstates of the Hamiltonian is exhausted by the set of states created by acting with primary or descendent operators inserted at the origin on the vacuum. The primary and descendent states considered in section 2.6 were shown to be examples of states that generate energy eigenstates, the argument above implies that these states form the full set energy eigenstates of the CFT. Solving a generic quantum mechanical system corresponds to identifying the spectrum of energy eigenstates[4], in a conformal field theory an equivalent role is played by the set of primary operators, drawing this conclusion requires one more ingredient.

2.7.3 Operator product expansion

Generally global states are not the practical objects to study in a local quantum field theory, the state-operator correspondence's greatest power is as a proof of principle. An important example is the operator product expansion, the equivalence between operators and states tells us that products of operators have to inherit certain properties of Hilbert spaces. One property of Hilbert spaces is that every element can be written as a convergent sum over a complete set of basis vectors.

[4] In principle at least, in practice decomposing a generic state into energy eigenstates can be an arbitrarily difficult problem.

Imagine two operators $O_1(z_1)$ and $O_2(z_2)$ inserted on the radial plane, these will generate a state on a circle centered at the origin that contains both operators

$$|\psi(\tau)\rangle = e^{-(\tau-\tau_2)L_0}O_2(z_2)e^{-(\tau_2-\tau_1)L_0}O_1(z_1)|0\rangle, \qquad \tau > \tau_2 > \tau_1. \tag{2.77}$$

This state $|\psi\rangle$ can be decomposed into a complete set of energy eigenstates $|\psi\rangle = \sum_i c_i|h_i\rangle$. In the previous section it was concluded that any energy eigenstate is generated by inserting a local operator with an appropriate scaling dimension at the origin. This suggests that any product of operators can be equivalently written as a sum over single operators

$$O_1(z_1)O_2(z_2) = \sum_i c_{12i}(z_1, z_2)O_{h_i}(0), \tag{2.78}$$

this expression can be brought to a more convenient form by demanding that the operator product expansion respects the translation invariance of theory. Perform two consecutive translations, first bring one of the operators to the origin, perform the operator product expansion and then translate back to the original positions

$$O_1(z_1)O_2(z_2) \to O_1(z_1 - z_2)O_2(0) = \sum_i c_{12i}(z_1 - z_2, 0)O_{h_i}(0)$$

$$\to O_1(z_1)O_2(z_2) = \sum_i c_{12i}(z_1, z_2)O_{h_i}(z_2) \tag{2.79}$$

$$= \sum_i \frac{c_{12i}}{(z_1 - z_2)^{h_1+h_2-h_i}}O_{h_i}(z_2) \tag{2.80}$$

exploiting translation invariance allows the right hand-side operators to be located at the point z_2 as long as the coordinate dependence of the coefficients $c_{12i}(z_1, z_2)$ is changed accordingly. This reflects the fact that choosing asymptotic past to be located at the origin was essentially arbitrary. The reason why this form is more convenient is that it allows one to easily determine how the the functions $c_{12i}(z_1, z_2)$ should respond to scale transformations. Once the location of the right hand-side operators has been fixed to z_2 the coordinate dependance of the coefficient functions is fixed by demanding that both sides posses the same scaling weight under scale transformations, this has been incorporated in the last equality. This is the operator product expansion of conformal field theory, the OPE for short.

The coefficients appearing on the far right side of (2.80) can be constrained further. By demanding consistency between the coordinate dependence of the three-point function of primary operators (2.69) and the orthogonality of conformal families the coefficients within a conformal family can be constrained

$$\langle O_1(z_1)O_2(z_2)O_3(z_3)\rangle = \frac{c_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_3-z_1)^{h_1+h_3-h_2}}, \tag{2.81}$$

while simultaneously

$$\begin{aligned}\langle O_1(z_1)O_2(z_2)O_3(z_3)\rangle &= \langle O_1(z_1)\sum_i c_{23i}(z_2-z_3)^{h_i-h_2-h_3}O_{h_i}(z_2)\rangle \\ &= \sum_{\{n_k\}} \frac{c_{123}(\{n_k\})}{(z_2-z_3)^{h_2+h_3-h_1-n_1-\ldots-n_k}}\langle O_1(z_1)L_{-n_1}\ldots L_{-n_k}O_1(z_3)\rangle. \end{aligned} \tag{2.82}$$

The sum on the bottom line runs over all ordered finite sequences of integers and represents the contribution of all descendant operators. This expression requires some explanation, the operator $O_1(z_1)$ in the above expression has the effect of projecting out the conformal family of O_1 from the OPE of O_2 and O_3. The scaling dimension of the desendant operators differ from the scaling dimension of the primary operator by integers, this allows the second line to be interpreted as a Taylor expansion of the three-point function around $z_2 \to z_3$. This justifies the notation $c_{123}(\{n_k\})$ for the OPE coefficients since the term corresponding to $n_i = 0$ matches the three-point function coefficient. Due to the fact that there are multiple states associated to a given level it requires more work to identify the other coefficients but they are in principle all fixed by the Virasoro algebra [46].

$$\begin{aligned} O_1(z_1)O_2(z_2) &= \sum_i \frac{c_{12i}}{(z_1-z_2)^{h_1+h_2-h_i}} \\ &\times \sum_{\{n_k\}} g(\{n_k\})\,(z_1-z_2)^{n_k+\ldots+n_1}[L_{-n_k},\ldots[L_{-n_1},O_{h_i}(z_2)]\ldots], \end{aligned} \tag{2.83}$$

in this last expression the first sum runs over all the conformal primaries of the CFT, all information that is fixed by conformal invariance has been explicitly filtered out in terms of the numbers of $g(\{n_k\})$, these numbers also depend on the scaling dimensions h_1 and h_2, but this has been suppressed in order to lighten the notation.

2.7.4 Conformal block decomposition

The operator product expansion is an extremely powerful tool, the only free parameters that remain unfixed by conformal invariance are the three-point function coefficients of primary operators. Given any higher point correlator, by successive application of the OPE it can be slowly boiled down to a large sum over two-point functions. In theory knowing all the OPE coefficients, or equivalently all three-point function coefficients, is enough to solve the entire CFT as it provides all dynamical information of all higher point correlators. In practice, resumming the contributions of all the states in a conformal representation is a very difficult problem.

The general procedure is best exemplified by the lowest non-trivial case, the four-point function of primary operators. By means of fractional linear transformation three out of four points can be mapped to three particular points of choice, the canonical choice is to map them to 0,1 and ∞. Inserting the OPE (2.83) into generic four-point function leads to

$$\langle O_1(z)O_2(0)O_3(1)O_4(\infty)\rangle = \sum_i \frac{c_{12i}c_{34i}}{z^{h_1+h_2-h_i}} \tag{2.84}$$

$$\times \lim_{w\to\infty} w^{2h} \sum_{\{n_k\}} \sum_{\{m_k\}} g(\{n_k\})g(\{m_k\})\; z^{n_1+\ldots+n_k} w^{m_1+\ldots+m_k} F(\{n_k\},\{m_k\},h), \tag{2.85}$$

where the group theoretical matrix components $F(\{n_k\},\{m_k\},h)$ are defined through

$$F(\{n_k\},\{m_k\},h) = \langle h|L_{n_1}...L_{n_k}L_{-m_k}...L_{-n_k}|h\rangle. \tag{2.86}$$

As can be seen from (2.85) the evaluation of the four-point function separates into a part that depends on the details of the theory, namely the sum over all primary operators and their three-point coefficients, and a a part that is entirely fixed by conformal invariance for a given conformal family. It is this latter part that is referred to as the conformal block.

For higher point functions the pairs by which the various operators are contracted through OPEs, called an OPE channel, is typically visualized by means of a diagram such as the one in figure 2.1

By the state-operator map the language of operators above can be equivalently phrased in terms of the states of the CFT, from which point of view the conformal block decomposition resembles inserting complete sets of states by means of the projection operators

$$\mathcal{P} = \sum_h P_h = \sum_h \sum_{\{n_k\},\{m_k\}} F^{-1}(\{n_k\},\{m_k\},h)\; L_{-n_k}...L_{-n_1}|h\rangle\langle h|L_{m_k}...L_{m_1}, \tag{2.87}$$

note of caution, due to the fact that the matrix F is not diagonal, the F^{-1} normalization factors in the above expression are given by the matrix elements

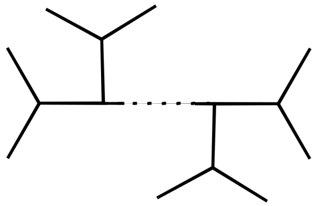


Figure 2.1: An example of the trivalent graphs that represent the OPE channels.

of the inverse matrix. The fact F is not diagonal ultimately follows from the fact that descendent states at the same level are not orthogonal. Inserting only the partial projectors P_h yields the conformal blocks constructed from the OPE above. It simultaneously explicitly reveals underlying principle by which the OPE was derived, through inserting a complete set of states.

The conformal blocks are entirely fixed by the Virasoro algebra, but for the general case no closed form expression exists for the conformal block. The way it is written above the series expression is essentially a polynomial expansion, unfortunately due to strong growth of the number of states at any descendent level computing the terms of the series quickly becomes unmanageable. If necessary, as an alternative, much more efficient recursion expressions for the conformal blocks are known [50].

2.8 Virasoro coadjoint orbits

This subchapter will contain a review of the Virasoro coadjoint orbits which make a pivotal appearance in chapters 5 and 6. First we will review that the Lie derivatives of orientation preserving diffeomorphisms of the circle (Diff(S^1)) form an algebra that in the Fourier basis gives the Witt algebra. As a result the Virasoro algebra has a geometrical meaning as the central extension of the Lie algebra of Diff(S^1). From this we will be able to read off that the CFT stress-tensor has a geometrical interpretation in terms of a quadratic differential field on a manifold whose tangent bundle is given by the (extended) family of infinitesimal elements of Diff(S^1). [124, 52, 65]

Since S^1 is 1-dimensional the algebra constructed out of their Lie derivatives is particularly simple. Given two vector fields on S^1, $V = f(\theta)\partial_\theta$ and $W = g(\theta)\partial_\theta$.

$$[W, V] = \mathcal{L}_W V = (f'g - g'f)\,\partial_\theta. \tag{2.88}$$

If we decompose the functions $f(\theta)$, $g(\theta)$ in Fourier modes

$$f(\theta) = \sum_n iL_n e^{i\theta(n+1)}, \tag{2.89}$$

and invert it we obtain the Witt algebra for the modes L_n

$$[L_n, L_m] = (n-m)L_{n+m}. \tag{2.90}$$

As is well-known, the Virasoro algebra is the central extension of the Witt algebra. As such we have to motivate a particular way to extend this algebra. The elements are easy, the basic elements of Witt algebra were the tangent vector fields on the circle parametrized by functions $f(\theta)$. To extend the elements we simply append to each function $f(\theta)$ a number to create a doublet $(f(\theta), t)$, these will be the elements of the extended algebra. The complication is to find a new bracket on this extended algebra, i.e given two doublets $(f(\theta), t_1)$, $g(\theta), t_2)$ what is

$$[(f(\theta), t_1), (g(\theta), t_2)] = ?$$

The standard CFT approach is to define the bracket through the 2d conformal Ward identity

$$\delta_\epsilon\langle X\rangle = -\frac{1}{2\pi i}\oint dz\, \epsilon(z)\langle T(z)X\rangle. \tag{2.91}$$

From this Ward identity we can associate the following charge operator to the infinitesimal conformal transformation $\epsilon(z)$

$$Q_\epsilon[T] = \frac{1}{2\pi}\oint dz\, \epsilon(z)T(z), \tag{2.92}$$

we associate the following Lie bracket to these conserved charges

$$[Q_{\epsilon_1}, Q_{\epsilon_2}][T] \equiv -Q_{\epsilon_2}[\delta_{\epsilon_1}T]. \tag{2.93}$$

We know the effect of the infinitesimal transformation on the stress tensor

$$\partial_\epsilon T(z) = -\frac{c}{12}\partial_z^3\epsilon(z) - 2T(z)\partial_z\epsilon(z) - \epsilon(z)\partial_z T(z), \tag{2.94}$$

hence we can work out the right-hand side of (2.93)

$$[Q_{\epsilon_1}, Q_{\epsilon_2}][T] = \frac{1}{2\pi}\oint dz\, \epsilon_2(z)\left(-2T(z)\partial_z\epsilon_1(z) - \epsilon_1(z)\partial_z T(z) - \frac{c}{12}\partial_z^3\epsilon_1(z)\right)$$

$$= \frac{1}{2\pi}\oint dz\, \left(\epsilon_1(z)\partial_z\epsilon_2(z) - \epsilon_2(z)\partial_z\epsilon_1(z)\right) T(z) - \frac{c}{24\pi}\oint dz\, \epsilon_2(z)\left(\partial_z^3\epsilon_1(z)\right)$$

$$= Q_{[\epsilon_1,\epsilon_2]}[T(z)] - \frac{c}{24\pi}\oint dz\, \epsilon_2(z)\left(\partial_z^3\epsilon_1(z)\right). \quad (2.95)$$

Note that the boundary term coming from the integration by parts vanishes due to the assumed single-valuedness of $T(z)$ and $\epsilon_i(z)$. This candidate Lie bracket does not look anti-symmetric yet, this can be fixed by performing two integrations by parts on the anomalous term

$$\oint dz\, \epsilon_2(z)\partial_z^3\epsilon_1(z) = -\oint dz\, \partial_z\epsilon_2(z)\partial_z^2\epsilon_1(z)$$

$$= -\oint dz\, \frac{1}{2}\left(\partial_z\epsilon_2(z)\partial_z^2\epsilon_1(z) + \partial_z\epsilon_2(z)\partial_z^2\epsilon_1(z)\right)$$

$$= \oint dz\, \frac{1}{2}\left(\partial_z^2\epsilon_2(z)\partial_z\epsilon_1(z) - \partial_z^2\epsilon_1(z)\partial_z\epsilon_2(z)\right). \quad (2.96)$$

This commutator algebra on the conformal charges of 2d CFT motivates the following bracket on the extended Witt algebra by transforming back to our coordinate system through $z = e^{i\theta}$. After some algebra we find

$$[(f(\theta), t_1), (g(\theta), t_2)] = \left([f(\theta), g(\theta)]\,,\; \frac{1}{48\pi}\int_0^{2\pi} d\theta\, e^{-2i\theta}\left(f'(\theta)g''(\theta) - g'(\theta)f''(\theta)\right)\right). \quad (2.97)$$

Write the space $(f(\theta)\partial_\theta, a)$ in terms of the following basis

$$L_n = \left(ie^{i(n+1)\theta}\partial_\theta, 0\right),\; Z = (0, i/2). \quad (2.98)$$

It is then easy to check that we find the usual Virasoro algebra (in a slightly unconventional form)

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{Z}{12}(n+1)^3\delta_{n+m,0}, \tag{2.99}$$
$$[L_n, Z] = 0. \tag{2.100}$$

The additional generator Z commutes with all other generators, hence this is a proper central extension of the Witt algebra.

2.8.1 Coadjoint action

In the last subsection it was reviewed how the Witt algebra can be viewed as the Lie bracket of vector field on S^1 and the Virasoro algebra as a central extensions of this algebra. To define the adjoint and coadjoint action on these vector fields we need to define an inner product. The vector fields $f(\theta)\partial_\theta$ and quadratic differential fields $b(\theta)d\theta^2$ possess a natural inner product

$$\langle b(\theta), f(\theta)\rangle = \int_0^{2\pi} d\theta\, f(\theta)b(\theta), \tag{2.101}$$

by simple Cartesian extension of this scalar product we can define an inner product on the extended space

$$\langle (b(\theta), a),\, (f(\theta), t)\rangle = \int_0^{2\pi} d\theta\, f(\theta)b(\theta) \,+\, at. \tag{2.102}$$

The adjoint representation is defined through the action of the vector fields onto themselves i.e.

$$ad_u(v) \equiv [u, v] \tag{2.103}$$

from this definition we can construct the adjoint action through the bracket (2.97). Inserting this into the inner product tells us the effect of the adjoint action

$$\begin{aligned}\langle (b,a),\, ad_{(g,x)}(f,y)\rangle &= \langle (b,a),\, \left(f'g - g'f,\, \frac{1}{48\pi}\int_0^{2\pi} d\theta\, e^{-2i\theta}(f'g'' - g'f'')\right)\rangle \\ &= \int_0^{2\pi} d\theta\, b(f'g - g'f) + \frac{a}{48\pi}\int_0^{2\pi} d\theta\, e^{-2i\theta}(f'g'' - g'f'').\end{aligned} \tag{2.104}$$

Given the adjoint action we define the coadjoint action as the unique operator on the dual space such that its effect on inner products is the same as that of the adjoint action[5], specifically

$$\langle ad^*_{(g,x)}(b,a),\,(f,y)\rangle \equiv -\langle (b,a),\,ad_{(g,x)}(f,y)\rangle. \tag{2.105}$$

Since the right hand-side is known we can compute what the effect of the coadjoint action is on a quadratic differential pair (b,a)

$$\begin{aligned}\langle ad^*_{(g,x)}(b,a),\,(f,y)\rangle &\equiv -\langle (b,a),\,ad_{(g,x)}(f,y)\rangle\\ &= -\int_0^{2\pi} d\theta\, bf'g - bg'f + \frac{a}{48\pi}e^{-2i\theta}(f'g''-g'f''),\end{aligned} \tag{2.106}$$

after performing some integrations by parts we obtain

$$\langle ad^*_{(g,x)}(b,a),\,(f,y)\rangle = \int_0^{2\pi} d\theta\,\left(-2bg' - b'g - \frac{a}{48\pi}e^{-2i\theta}\left(2g''' - 6ig'' - 4g'\right)\right) f, \tag{2.107}$$

from which we can read off the effect of the coadjoint action

$$ad^*_{(g,x)}(b,a) = \left(2bg' + b'g + \frac{a}{48\pi}e^{-2i\theta}\left(2g''' - 6ig'' - 4g'\right),\, 0\right). \tag{2.108}$$

This expression looks unfamiliar but if we apply the change of variables $z = e^{i\theta}$ on the right hand-side of (2.107) we obtain

$$\langle ad^*_{(g,x)}(b,a),\,(f,y)\rangle = \oint dz\left(2bg' + b'g - \frac{a}{24\pi}g'''\right) f \tag{2.109}$$

hence in z-coordinates the coadjoint action takes the form of the variation of the 2d CFT stress-tensor under infinitesimal conformal transformations generated by g. This justifies the description maintained throughout the body of the text, that the sets of all stress-energy tensors that are connected to each other through single-valued conformal transformations is isomorphic to the coadjoint orbits generated from a quadratic differential field on the extended space by acting with the coadjoint action.

2.8.2 Classifying orbits

In the previous subsection it was shown that due to the transformation rule of the stress tensor it naturally lives inside a Virasoro coadjoint orbit, i.e.

[5] Up to a sign that is, this sign convention ensure that the combination $\langle ad^*_{(g,x)}(b,a),\,(f,y)\rangle + \langle (b,a),\,ad_{(g,x)}(f,y)\rangle$ vanishes.

the space of all quadratic differentials that are continuously connected to some reference quadratic differential field. These coadjoint orbits possess a natural manifold structure [51], where we can think of the coadjoint action parametrized by a pair (f, x) as (almost) the tangent space at the reference point (b, a). This is not entirely accurate, there are possibly also pairs (h, y) that leave the reference point (b, a) invariant, these have to be modded out. Define the central extension of the algebra of vector fields on S^1 as $\overline{diff(S^1)}$ and G_b as the the algebra spanned by the vector fields (h, y) such that $ad^*_{(h,y)}(b, a) = 0$ specifically

$$G_b = \left\{ h \in \overline{diff(S^1)} \mid ad^*_h(b, a) = 0 \right\}. \tag{2.110}$$

In that case the tangent space of the coadjoint orbit at the reference point (b, a) is given by $\overline{diff(S^1)}/H_b$. This tangent space can be integrated to a full manifold designated by $\overline{Diff(S^1)}/H_b$, these will be identified as the coadjoint orbits. As a result the coadjoint orbits are classified by their reference point and their stabilizer algebra. Turning back to the stabilizer algebra, since the coadjoint action is given by

$$ad^*_{(f,x)}(b, a) = \left(2bf' + b'f + \frac{a}{48\pi} e^{-2i\theta} \left(2f''' - 6if'' - 4f' \right), 0 \right). \tag{2.111}$$

we know that given a fixed point (b, a) the defining equation for the stabilizer vectors h is

$$2bh' + b'h + \frac{a}{48\pi} e^{-2i\theta} \left(2h''' - 6ih'' - 4h' \right) = 0, \tag{2.112}$$

or after a change of variables $z = \exp(i\theta)$

$$2bh' + b'h - \frac{a}{24\pi} h''' = 0. \tag{2.113}$$

As a first step towards the well-studied [124] connection between Virasoro coadjoint orbits and the differential equations classified as Hill's equation we note that given a basis of solutions $\psi_1(z)$ and $\psi_2(z)$ to the linear ODE

$$\psi''(z) - b(z)\psi(z) = 0, \tag{2.114}$$

the independent solutions of (2.113) will be given by

$$\psi_1(z)^2, \quad \psi_2(z)^2, \quad \psi_1(z)\psi_2(z). \tag{2.115}$$

There is one additional complication, the condition (2.113) is a local condition, there is no guarantee that the local solutions (2.115) respect the periodicity

Orbit	Stabilizer algebra generators
$\overline{Diff(S^1)}/S^1$	l_0
$\overline{Diff(S^1)}/PSL^{(n)(2,\mathbb{R})}$	l_0, l_n, l_{-n}
$\overline{Diff(S^1)}/T_{n,\Delta}$	$f(\theta)$, where f is a vector field with $2n$ simple roots
$\overline{Diff(S^1)}/\tilde{T}_{n,\pm}$	$f(\theta)$, where f is a vector field with n double roots

Table 2.1: The classification of Virasoro coadjoint orbits purely in terms of group manifolds, i.e. without their reference points

condition of the circle. In a particularly impressive proof in [54] it is shown that the number of global stabilizing vector fields of a given reference point is always either 1 or 3.

The classes of solutions to (2.113) have been classified [54], the resulting (either 1 or 3 dimensional) stabilizer groups that can be uplifted from the solutions are typically listed as [124]

$$S^1, \quad , PSL^{(n)}(2,\mathbb{R}), \quad T_{n,\Delta}, \quad \tilde{T}_{n,\pm}. \tag{2.116}$$

Respectively, S^1 forms the set of rigid rotations of the circle, $PSL^{(n)}(2,\mathbb{R})$ is the group generated from the subalgebra of $\overline{diff(S^1)}$ spanned by (L_0, L_{-n}, L_n). $T_{n,\Delta}$ and $\tilde{T}_{n,\pm}$ are one-dimensional groups whose Lie algebras consist of vector fields on S^1 with respectively $2n$ simple zeros or n double zeros, since the number of zeros is an orbit invariant The list of these group manifolds that form the various Virasoro coadjoint orbits has been summarized in table 2.1 [65].

2.8.3 Hill's equation, Virasoro coadjoint orbits and $SL(2,\mathbb{R})$ conjugacy classes

So far this concludes the classification as presented in [54], the group manifolds formed by the coadjoint orbits form somewhat intangible structures. In this section we will follow [124] and link the Virasoro coadjoint orbits to something that is of direct physical interest for our purposes; the conjugacy classes of the monodromy matrices of the solutions $\psi(z)$ of

$$\psi''(z) + \frac{6}{c}T(z)\psi(z) = 0, \tag{2.117}$$

around the unit circle. The information of the coadjoint orbit enters the ODE through $T(z)$ which is related to the earlier quadratic differential fields $b(\theta)d\theta^2$

through $T(z) = b(x(z))$ with $z = e^{i\theta}$. In order to directly connect to application in 2d CFT described in section 2 we identify $T(z)$ with the stress tensor expectation value of the CFT used throughout the body of the text, in that vein c designates the central charge of theory. Being a second order linear differential equation, the solution space of (2.117) is spanned by a basis of solutions of the form $\boldsymbol{\psi}(z) = (\psi_1(z), \psi_2(z))^T$, the monodromy matrix $\boldsymbol{\psi}(z)$ around the unit circle $M_\psi \in SL(2, \mathbb{R})$ is given by

$$\boldsymbol{\psi}\left(e^{2\pi i} z\right) = M_\psi \boldsymbol{\psi}(z) \tag{2.118}$$

The infinitesimal transformation under a vector field (f, y) was given in (2.109) it is known that this expression integrates to the transformation rule

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12} S[z, w], \tag{2.119}$$

where $w(z)$ is an element of $Diff(S^1)$ analytically continued to the complex plane and $S[z, w]$ is the Schwarzian derivative defined as

$$S[z, w] = \frac{z'''(w)}{z'(w)} - \frac{3}{2}\left(\frac{z''(w)}{z'(w)}\right). \tag{2.120}$$

It is straightforward to check that if $\psi(z)$ solves (2.117) that under the replacement $T(z) \to T(w)$ as in (2.119) that (2.117) is solved by the replacement $\psi(z) \to \phi(w) = \frac{1}{\sqrt{z'(w)}} \psi(z(w))$. Since $w(z)$ is by construction single-valued along the unit circle we can conclude that the monodromy matrix M_ψ is invariant under the transformations $w(z)$. Of course in constructing the basis of solutions $\boldsymbol{\psi}(z)$ there exists a freedom in choice of basis, a basis tranformation $\boldsymbol{\psi}_A(z) = A\boldsymbol{\psi}(z)$ has the effect of conjugating the monodromy matrix

$$M_{\psi_A} = A^{-1} M_\psi A. \tag{2.121}$$

This establishes an important fact namely that the conjugacy class of the monodromy matrix M_ϕ is an orbit invariant with respect with the entire Virasoro orbit in which $T(z)$ is contained.

Since every Virasoro orbit maps to a conjugacy class of $SL(2, \mathbb{R})$ and since these conjugacy classes are known, the problem is to link the right orbit to the right conjugacy class. The strategy is to match the respective stabilizer subgroups. Take $\alpha(z)$ to be an element of the stabilizer group of $T(z)$, i.e.

$$T(z) = \left(\frac{dz}{d\alpha}\right)^2 T(z) + \frac{c}{12} S[z, \alpha]. \tag{2.122}$$

The fact that $\alpha(z)$ leaves the differential equation invariant but changes the solution implies that the new solutions are related to the old ones by linear transformation

$$\frac{1}{\sqrt{\alpha'(z)}}\boldsymbol{\psi}(\alpha(z)) = \gamma(\alpha)\boldsymbol{\psi}(z), \tag{2.123}$$

the left hand side leaves M_ψ invariant whereas the right-hand side affects M_ψ through conjugation as a result the matrix $\gamma(\alpha)$ has to satisfy

$$M_\psi = \gamma(\alpha)^{-1} M_\psi \gamma(\alpha). \tag{2.124}$$

As a mapping $\gamma(\alpha)$ has the property of mapping of mapping an element of the stabilizer group to a matrix that leaves the matrix M_ψ invariant under conjugation, this set of matrices forms a group $G[M_\psi]$ and $\gamma(\alpha)$ forms a homomorhpism from the stabilizer group to $G[M_\psi]$. This led the authors of [124] to the following strategy: take a conjugacy class of $SL(2,\mathbb{R})$, pick an element of that class to act as a representative, compute the set of matrices that leave the representative invariant under conjugation. As a final, most computationally challenging, task find a $T(z)$ such that a basis of solutions to (2.117) exists such that it's monodromy around the unit circle is given by the representative matrix. As a result one will have found the set of representatives and stabilizer groups hence classifying all Virasoro coadjoint orbits.

The set of eigenvalues of an element of $SL(2,\mathbb{R})$ are invariant under conjugation, the $\det(M_\psi) = 1$ constraint restricts the two eigenvalues of M_ψ to be each others inverse, Floquet's theorem further restricts the relevant classes to either be purely real or pure phase with cross-over points where both eigenvalues are either 1 or both -1. The orbits with eigenvalues falling in these categories are respectively designated as hyperbolic, elliptic and parabolic or exceptional orbits. By constructing explicit examples of stress tensors $T(z)$ whose associated solutions have a monodromy within these conjugacy classes the authors of [124] list the following classes of Virasoro orbits

$$\mathcal{B}_n(b), \quad \mathcal{C}(\nu), \quad \mathcal{P}_n^{\pm}, \quad \mathcal{E}_n \tag{2.125}$$

these respectively correspond hyperbolic, elliptic, parabolic and exceptional orbits, but they can have different stabilizer algebras corresponding to the affixes n and $\pm$ the real arguments b and ν parametrize the set of representative stress tensors.

2.8.4 Relevant qualitative features of the individual orbits

As we shall see not all the orbits mentioned in the previous section are of direct physical interest though. To each stress tensor we an associate an energy expectation value through the conserved charge L_0

$$L_0 = \frac{1}{2\pi i}\oint dz\, zT(z), \tag{2.126}$$

while within each orbit this number is unbounded from above, there are only a few orbits for which this number bounded from below. Since we are considering unitary CFTs the energy expectation value on the entire spectrum of states is bounded from below by the energy expectation value of the vacuum state (in any conformal frame). Therefore one can dismiss any orbit without a lower bound as incapable of being associated to a physical state of the CFT spectrum. The full list of orbits which possess a lower bound on their energy is given by

$$\mathcal{B}_0(b), \quad \mathcal{C}(\nu)\ \ (0<\nu<1), \quad \mathcal{P}_0^+, \quad \mathcal{E}_1, \quad \mathcal{P}_1^-. \tag{2.127}$$

In addition some of these orbits have the special property that they contain constant representatives[6] these orbits and their representatives are given by

$$\begin{array}{ll} \mathcal{E}_1 & T(z) = 0 \\ \mathcal{C}_\nu\ \ (0<\nu<1) & T(z) = \frac{c}{24}(1-\nu^2)\frac{1}{z^2}, \\ \mathcal{P}_0^+ & T(z) = \frac{c}{24}\frac{1}{z^2}, \\ \mathcal{B}_0(b) & T(z) = \frac{c}{24}(1+b^2)\frac{1}{z^2}. \end{array}$$

The representative functions $T(z)$ have the form of the stress-tensor expectation values on a state given by conformal primary inserted at the origin acting on the vacuum with a scaling dimension given by the numerical prefactor of $T(z)$. This also serves as a first indicator that something seems to change at the BTZ mass threshold $c/24$. Some of the details of the classification have been summarized in table 2.2.

[6]The name is due to fact that in the coordinate x of [124],[54] these representatives are in fact constants, for the 2d CFT purposes of this note it is convenient to immediately express these representative stress tensors in the z-coordinate of the radial plane, where constant representative is perhaps a bit of a misnomer. Hence in the remainder of the text they are instead referred to as primary representatives.

Orbit	Stabilizer	$SL(2,\mathbb{R})$ class	Bounded L_0	Primary repr.
$\mathcal{B}_0(b)$	S^1	hyperbolic	yes	yes
$\mathcal{B}_{n>0}(b)$	$T_{n,\Delta}$	hyperbolic	no	no
$\mathcal{C}(\nu)$ $(0<\nu<1)$	S^1	elliptic	yes	yes
$\mathcal{C}(\nu)$ otherwise	S^1	elliptic	no	yes
$\mathcal{P}_0^+$	S^1	parabolic	yes	yes
$\mathcal{P}_1^-$	$\tilde{T}_{1,-}$	parabolic	yes	no
$\mathcal{P}_n^\pm$	$\tilde{T}_{n,\mp}$	parabolic	no	no
$\mathcal{E}_1$	$PSL(2,\mathbb{R})$	exceptional	yes	yes
$\mathcal{E}_{n\neq 1}$	$PSL^{(n)}(2,\mathbb{R})$	exceptional	no	no

Table 2.2: Relevant features of the Virasoro coadjoint orbits as classified in [124].

Chapter 3

Gravity in three dimensions

The force of gravity separates itself from the rest of the forces observed in nature by eluding a quantum mechanical description in terms of a quantum field theory. The theory of general relativity forms a beautiful minimalistic framework that describes gravity at the largest scales. This minimalism is reflected in the fact that when written as an action principle it possesses, in conventional notation, the most succinctly written action in the contemporary theoretical physics canon

$$S_{EH} = \int d^d x \sqrt{-g} \left(\frac{1}{16\pi G}(R - 2\Lambda) + \mathcal{L}_M \right) . \tag{3.1}$$

This can be regarded as the lowest scale effective field theory of a theory whose dynamical variables are the metric field components and possesses a gauge invariance given by the coordinate transformations, additionally the source of the metric fluctuations is given by the canonical stress tensor of the matter sector $\mathcal{L}_M$. While, say, the physics of classical mechanics is generally also insensitive to coordinate transformations, what separates general relativity is that not just the physics but the action and equations of motion themselves remain *invariant* under coordinate transformations.

General relativity has famously been tested to great success in what are considered the four great tests of general relativity. These include the matching of the relativistic correction to the orbit of Mercury to its perihelion shift, the observed gravitational lensing during a solar eclipse, and precision measurements of the lapse of time at different altitudes above the surface of the earth

[56]. A fourth later test was proposed in 1968, the Shapiro time delay measured when bouncing radar signals of the surface Venus and Mercury [57]. Much more recently the LIGO collaboration has been able to observe gravitational waves predicted by general relativity. The extreme precision required to perform this measurement and its perfect coincidence with the predictions of numerical relativity put general relativity in the laudable position of currently best tested physical theory [59].

While the success of general relativity at the classical level is by now undeniable, the unification with quantum mechanics has been problematic. The relevant dimensionful length scale constructed out of the natural constants of a potential quantum gravity theory would be the Planck scale

$$l_{pl} = \sqrt{\frac{\hbar G}{c^3}}, \tag{3.2}$$

if general relativity is considered as an effective field theory, then it is in fact the best effective field theory currently at our disposal as any irrelevant terms added to the Einstein-Hilbert action will be weighted by the Planck length [3]. Performing an expansion of the metric around flat space $g_{\mu\nu} = \eta_{\mu\nu} + \frac{4\sqrt{\pi G}}{c^2} h_{\mu\nu}$ and expanding the Einstein-Hilbert action in powers of the metric deviation $h_{\mu\nu}$ yields an action with interaction terms weighted by a dimensionful coupling constant. This is often argued as a signature of non-renormalizability of gravity, the argument being that the number of counter terms generated from a renormalization scheme would never terminate.

While this is a strong argument in disfavor of an orthodox quantization scheme for gravity, there also exists strong set of fundamental reasons suggesting that at the microscopic level quantum gravity is fundamentally incompatible with the basic tenets of quantum field theory. The gauge invariance of general relativity, diffeomorphism invariance, suggests that all sensible gauge invariant observables should be inherently non-local, which is incongruous with the locality of quantum field theory interactions [55]. Secondly in more conventional gauge theories the local field transformations generated by the gauge transformations push the field configurations into a spatial direction. The diffeomorphisms of general relatvity on the other hand have the power to shift points into a time-like direction. Reconciling this feature with the unitary time-evolution tells us that quantum gravity will not be a conventional canonical quantization of the classical general relativity degrees of freedom. This sometimes referred to as the problem of time [55, 61].

3.1 Asymptotically AdS$_3$ gravity

Just like in the previous chapter, where 2d conformal field theory separated itself from conformal field theory in higher dimensions, gravity in 3d takes on a somewhat different guise with respect to higher dimensional gravity. Three is the minimal number of dimensions where the dynamics of gravity can be written in terms of a local action, however while the action is still local its stationary points are highly restricted. The dynamics of spacetime are encoded within the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{3.3}$$

these equations of motion are same in all dimensions, but in three dimensions they are far more constraining. This is due to the fact that in three dimensions the number of functions necessary to specify a geometry is reduced significantly, this is reflected in the fact that in 3d the Riemann tensor is completely fixed by the Ricci tensor

$$R^{\mu\nu}_{\sigma\rho} = -\frac{1}{g}\epsilon^{\mu\nu\alpha}\epsilon_{\sigma\rho\beta}g^{\beta\lambda}\left(R_{\lambda\sigma} - \frac{1}{2}g_{\lambda\sigma}R\right). \tag{3.4}$$

One can make use of the following identity

$$\epsilon^{\mu\nu\alpha}\epsilon_{\rho\sigma\alpha} = g\left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho}\delta^{\mu}_{\rho}\right), \tag{3.5}$$

these expressions can be found in a large body of references, an example includes [60]. After imposing the vacuum Einstein field equations on the expression for the Riemann tensor above one obtains

$$R_{\mu\nu\rho\sigma} = \Lambda\left(g_{\mu\nu}g_{\rho\sigma} - g_{\nu\rho}g_{\mu\sigma}\right), \tag{3.6}$$

contracting the indices of the Riemann tensor above leads to a constant value for the Ricci scalar

$$R = 6\Lambda, \tag{3.7}$$

hence in three dimensions all solutions to the Einstein field equations will possess a constant curvature fixed by the cosmological constant. Secondly this means that after imposing the vacuum Einstein field equations the Riemann tensor of this space coincides with those of the general form

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}\left(g_{\mu\nu}g_{\rho\sigma} - g_{\nu\rho}g_{\mu\sigma}\right) \tag{3.8}$$

This expression can be recognized as the equivalent statement that the spacetime manifold is a maximally symmetric space [12]. All maximally symmetric spaces have been classified up to diffeomorphism and if we assume the cosmological constant Λ to take on negative values then our spacetime has to be given by anti-de Sitter space up to diffeomorhpisms and global identifications. What this tells us is that given *any* open set U and any solution to the vacuum Einstein field equations with negative cosmological constant $g_{\mu\nu}$ there exists a gauge transformation that brings brings the metric $g_{\mu\nu}$ to the AdS$_3$ metric.

There are two commonly used coordinate systems to describe Anti-de Sitter space, the first consists of the Poincaré coordinates given by the line element

$$ds^2 = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2 + \frac{r^2}{R^2}d\phi^2. \tag{3.9}$$

In principle the domain of this metric field, called the Poincaré wedge, can be defined as AdS$_3$, but it can be expanded into a larger space by considering the universal cover of the Poincaré wedge, this universal cover only needs a single coordinate chart to be described in full

$$ds^2 = R^2\left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\phi^2\right), \tag{3.10}$$

this coordinate system is hence appropriately called the global coordinate system [10]. The parameter R_{AdS} that appears in both these metrics is the relevant characteristic length scale of AdS, the AdS radius[1], by dimensional analysis or substitution into the Einstein field equations it can be found to be related to Λ through $\Lambda = -R_{AdS}^{-2}$.

To gain some intuition into this class of spaces it is useful to study geodesic motion of massive particles. Consider radial motion, the proper time constraint $1 = g_{\mu\nu}\dot{x}_\mu\dot{x}_\nu$ combined with the effect that this metric is independent of time ensures that a massive particle following geodesic motion experiences an effective potential of the form

$$V_{\text{eff}} = m\frac{r^2}{R^2}, \tag{3.11}$$

where m is the mass of the particle. This can be recognized as the potential energy of a harmonic oscillator, hence all the intuition of harmonic oscillator can be carried over into the radial motion of massive particles in AdS. For one all particles feel an attractive force towards the center of AdS and a released particle will oscillate around the origin. Naively this would suggest that AdS

[1] From this point onwards R will always indicate the AdS radius as opposed to the Ricci scalar.

is not a homogenous space as the origin appears to play a special role. To counteract this false intuition imagine an observer displaced from the origin. From the perspective of an outside observer this observer would free-fall towards the center, but due to the equivalence principle there exists another coordinate frame from which the falling observer is at rest and the rest of the matter content is moving with respect to the falling observer. Carefully working out this coordinate transformation one finds that the new coordinate system is just an AdS coordinate system with the origin centered around the new observer.

The fact that any solution of the Einstein field equations can be brought back to the global coordinate system by means of a gauge transformation (in any open set at least) suggests that there are no measurable gravitational wave solutions. If a 3d quantum theory of gravity would posses on-shell gravitons than the classical low energy limit should posses solutions corresponding to coherent motion of many such gravitons. This suggests that a quantum theory of pure 3d gravity has no on-shell gravitons, which puts the theory at serious risk of being trivial. Pure 3d gravity with a negative cosmological constant avoids this fate in two inter-connected ways

- Any gauge theory contains a space of surface charges generated by the asymptotic gauge transformations, the gauge transformations that do not decay to zero fast enough at infinity but do preserve the field boundary conditions [62]. These surface degrees of freedom contribute their own dynamics that shares features with the 2d CFTs of the last chapter. [18]

- The fact that the solutions to the vacuum EFEs have to be related by a gauge transformation to AdS$_3$ in any open set leaves the door open for non-trivial solutions that differ from AdS$_3$ by global identifications. This opens up the possibility for conical defect geometries and, more suprisingly, black hole solutions [63, 64].

These two points will be the topic of the next two sections. As a last mention, this last discussion is based on the vacuum Einstein field equations, but almost all conclusions persist after adding matter. Matter is coupled to these equations by means of a local stress tensor field, away from the location of matter it remains true that no gravitational waves propagate. After some matter state has equilibriated, including the effects of backreaction, the resulting geometry should be static and when away from matter sources gauge equivalent to AdS$_3$. As we will see when we discuss conical geometries, geometries containing non-trivial global identifications can be interpreted as being sourced by matter fields.

3.2 Asymptotic symmetries and the Virasoro algebra

Gauge freedom arises from an over excessive description for the set of true physical states. This typically happens by packing the physical degrees of freedom into spacetime symmetry representations that are too large in order to make the spacetime symmetry manifest. It is expected that asymptotic AdS_3 gravity has an equivalent description in terms of a 2d conformal field theory. When it is said that is expected that the partition function of the gravity side matches the partition function of the CFT theory, this is restrained to gauge invariant states. For this reason the gauge symmetries on both sides of the duality do not match.

The same does not hold true for the global symmetries of the theory. In the previous chapter it was demonstrated that the presence of conformal symmetry in two dimensions leads to an infinite set of measurable charges. These charges should have an equal counterpart in the gravitational theory. These surface charges of AdS_3 gravity were observed in the seminal work of Brown and Henneaux [18] predating the AdS/CFT correspondence by over a decade. These charges correspond to the surface charges induced by diffeomorphisms that do not die out rapidly enough at infinity. As a consequence the resulting central charge of the Poisson bracket algebra of these charges gives a hint as to what CFTs can be expected to have semi-classical gravity duals. Secondly the Virasoro coadjoint orbits that were covered extensively in the last chapter will make an appearance when classifying inequivalent solutions of the Einstein field equations.

The metric field can only potentially deviate from that of empty AdS_3 at points in spacetime where the stress tensor takes on non-zero values, assume that the distribution of matter dilutes at $r \to \infty$. The strategy explored by [18] is to study geometries that in the $r \to \infty$ limit asymptotically approach anti-de Sitter space. The art is to choose the boundary conditions such that the spectrum of charges at least contain a set of charges corresponding to the currents of the AdS_3 isometry algebra of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. This inspired them study the following class of metric fields as a guiding principle

$$ds^2 = -\left(\frac{r^2}{R_{AdS}^2} + \alpha^2\right) dt^2 + 2A\alpha dt d\phi + \left(\frac{r^2 - A^2}{R_{AdS}^2} + \alpha^2)\right)^{-1} dr^2 + (r^2 - A^2) d\phi^2, \tag{3.12}$$

this class of metrics was inspired by the conical geometry solutions, here α parametrizes the size of the missing wedge of the angular coordinate and A parametrizes the 'time-skip' experienced when traveling along angular coordinate. By acting on this class of metrics with the usual AdS isometries, it is shown that if at $r \to \infty$ the following boundary conditions are imposed,

$$\begin{aligned} g_{tt} &= -\frac{r^2}{R_{AdS}^2} + \mathcal{O}(1), \\ g_{tr} &= \mathcal{O}(r^{-3}), \\ g_{t\phi} &= \mathcal{O}(1), \\ g_{rr} &= \frac{R_{AdS}^2}{r^2} + \mathcal{O}(r^{-4}), \\ g_{r\phi} &= \mathcal{O}(r^{-3}), \\ g_{\phi\phi} &= r^2 + \mathcal{O}(1), \end{aligned} \tag{3.13}$$

that the usual set of AdS$_3$ isometries is contained within the set of asymptotic symmetries that preserve the boundary conditions. These boundary conditions are often referred to as the Brown-Henneaux boundary condtions [69, 65, 70, 71][18]. While designed to allow the AdS$_3$ isometries they allow for a larger set of vector fields that preserve the boundary conditions. In this set of coordinates the algebra of boundary condition preserving vector fields is quite involved. With the benefit of hindsight the analysis is simplified in the lightcone coordinates $x = t/R + \phi$, $\bar{x} = t/R - \phi$, this has the added advantage added advantage of making it easier to identify the boundary of Lorentzian AdS$_3$ with the spacetime of Lorentzian 2d CFT. In lightcone coordinates the boundary conditions can equivalently be written as

$$g_{x\bar{x}} = -\frac{r}{2} + R_{AdS}^2 \gamma_{x\bar{x}}(x, \bar{x}) + O(1/r), \tag{3.14}$$

$$g_{xx} = R^2 \gamma_{xx}(x, \bar{x}) + O(1/r) \qquad \text{and } (x \leftrightarrow \bar{x}), \tag{3.15}$$

$$g_{xr} = R^4 \frac{\gamma_{xr}(x, \bar{x})}{r^3} + O(1/r^4) \qquad \text{and } (x \leftrightarrow \bar{x}), \tag{3.16}$$

$$g_{rr} = \frac{R^2}{r^2} + l^4 \frac{\gamma_{rr}(x, \bar{x})}{r^4} + O(1/r^5). \tag{3.17}$$

Where the undetermined functions are generic functions of the boundary surface coordinates [65]. The components of the vector fields that preserve these boundary conditions are given by

$$\zeta^x = \xi(x) + \frac{R^2}{2r^2}\partial_{\bar{x}}^2\bar{\xi}(\bar{x}) + O(1/r^4), \tag{3.18}$$

$$\zeta^{\bar{x}} = \bar{\xi}(\bar{x}) + \frac{R^2}{2x^2}\partial_x^2\xi(x) + O(1/r^4), \tag{3.19}$$

$$\zeta^r = -\frac{r}{2}(\partial_x\xi(x) + \partial_{\bar{x}}\bar{\xi}(\bar{x})) + O(1/r). \tag{3.20}$$

By decomposing the functions $\xi(x)$ and $\bar{\xi}(\bar{x})$ into Fourier modes and by raising and lowering indices with the metric above one can in principle construct the Lie bracket algebra of these vector fields. This is a rather involved computation and one that is not particularly insightful, fortunately once again the benefit of hindsight provides a more efficient strategy. In [66] it was noted that the most general metric that satisfies the boundary conditions (3.17) in the Fefferman-Graham gauge is given by

$$ds^2 = \frac{R^2}{r^2}dr^2 - \left(rdx - \frac{R^2}{r}\bar{L}(\bar{x})d\bar{x}\right)\left(rd\bar{x} - \frac{R^2}{r}L(x)dx\right). \tag{3.21}$$

This family of metrics is parametrized by two analytic functions $L(x)$, $\bar{L}(\bar{x})$ with a periodicity reflecting the angular identification $\phi \sim \phi + 2\pi$. This is the most general metric in the sense that any asymptotic transformation preserves the overall form of the metric and only changes the functions $L(x)$ and $\bar{L}(\bar{x})$. This can be phrased more explicitly as

$$g_{\mu\nu}(L(x), \bar{L}(\bar{x})) + \delta_{\zeta,\bar{\zeta}}g_{\mu\nu}(L(x), \bar{L}(\bar{x})) + \mathcal{O}(\xi^2, \bar{\xi}^2) = g_{\mu\nu}(\delta_\zeta L(x), \delta_{\bar{\zeta}}\bar{L}(\bar{x})), \tag{3.22}$$

where

$$\delta_\zeta L(x) = 2L(x)\partial_x\xi(x) + \xi(x)\partial_x L(x) - \frac{1}{2}\partial_x^3\xi(x), \tag{3.23}$$

with a similar expression for $\bar{L}(\bar{x})$ in terms of $\bar{\xi}(\bar{x})$. The first clue to the connection to the Virasoro algebra is formed by the fact that the transformation rule for $L(x)$ is of the same form as the transformation rule for the CFT stress tensor $T(z)$ under infinitesimal conformal transformations (2.48). This implies that the parameter functions $L(x)$ and its right-moving counterpart are elements of Virasoro coadjoint orbits that were discussed in great detail in the last chapter.

To find the conserved charges of these boundary condition preserving vector field one can apply the decomposition formula of Iyers-Wald surface charges [73]. This formula states that in n dimensions the surface charge density $(n-2)$-form of a generic diffeomorphism invariant Lagrangian theory can be written in the form

$$\mathbf{Q} = \mathbf{W}_c(\phi)\zeta^c + \mathbf{X}^{cd}(\phi)\nabla_{[c}\zeta_{d]}, \tag{3.24}$$

where

$$\mathbf{X}^{cd}_{c_3\ldots c_n} = -\frac{\delta \mathcal{L}}{\delta R_{abcd}}\epsilon_{abc_3\ldots c_n}. \tag{3.25}$$

Explicit calculation of the symplectic potential current of the Einstein-Hilbert action suggests that $\mathbf{W}_c = 0$, see e.g. page 12 of [73] for details. Setting the number of dimensions to three and applying this formula to the Einstein-Hilbert action (3.1) gives the following charge density 1-form

$$\mathbf{Q}_\mu dx^\mu = -\frac{1}{16\pi G}\epsilon_{\alpha\beta\mu}dx^\mu\nabla^\alpha\zeta^\beta. \tag{3.26}$$

The actual charge is found by integrating this charge density along a circle on the boundary. Writing out the covariant derivative in terms of the affine connections coefficients and ignoring all the full derivative terms and the parts of $\nabla^\alpha\zeta^\beta$ that are symmetric in the indices leads to

$$Q_\zeta = -\frac{1}{16\pi G}\int_{S^1} dx^\mu\,\epsilon_{\alpha\beta\mu}g^{\beta\nu}\zeta^\sigma\partial^\alpha g_{\nu\sigma}. \tag{3.27}$$

Furthermore inserting the Bañados metric (3.21) and taking the limit $r\to\infty$ reduces this integrated charge to the expression

$$\begin{aligned} Q_\zeta = {} & \frac{R}{8\pi G}\int_0^{2\pi} 2\bar{\xi}(x)r^{-1} - \xi(x)L(x)r^{-3}dx \\ & + \frac{R}{8\pi G}\int_0^{2\pi} 2\xi(\bar{x})r^{-1} - \bar{\xi}(\bar{x})\bar{L}(\bar{x})r^{-3}d\bar{x} + \mathcal{O}(r^{-4}), \end{aligned} \tag{3.28}$$

The presence of the leading term of order $1/r$ is in a sense fictional, it is independent of the metric parameters $L(x)$ and $\bar{L}(\bar{x})$ and captures a universal vacuum piece of the charge. This term is related to the divergent nature of the Einstein-Hilbert action, when performing a proper treatment that involves renormalizing the gravitational action by adding counter-terms in order to render it finite this term disappears from the charge [76, 77]. In such a case these terms leading terms drop out and the charge reduces to the simpler formula

$$Q_\zeta = -\frac{R}{8\pi G}\left(\int_0^{2\pi} dx\,\xi(x)L(x) + \int_0^{2\pi} d\bar{x}\,\bar{\xi}(\bar{x})\bar{L}(\bar{x})\right) \tag{3.29}$$

The similarity of this expression to the conformal charge version (2.44) of the previous chapter is apparent if we transform (2.44) from complex radial plane coordinates to real lightcone coordinates through $z = \exp(ix)$, $\bar{z} = \exp(i\bar{x})$

$$Q_\epsilon = -\frac{1}{2\pi}\int_0^{2\pi} dx\, e^{-ix}\epsilon\left(e^{ix}\right)\left(T(x) - \frac{c}{24}\right). \tag{3.30}$$

Note the similarities of these two expressions for the charges if we split the gravitational surface charge into holomorphic and anti-holomorphic parts[2]. In fact if we separate the surface charges into holomorphic and anti-holomorphic charges it is convenient to make the identifications

$$\xi(x) \leftrightarrow e^{-ix}\epsilon\left(e^{ix}\right), \qquad \bar{\xi}(\bar{x}) \leftrightarrow e^{-i\bar{x}}\epsilon\left(e^{i\bar{x}}\right), \tag{3.31}$$

$$\frac{R}{4G}L(x) \leftrightarrow T(x), \qquad \frac{R}{4G}\bar{L}(\bar{x}) \leftrightarrow \bar{T}(\bar{x}), \tag{3.32}$$

These identifications imply that the entire analysis of chapter 2 can at this point be repeated and the resulting surface charge algebra should again be the Virasoro algebra. The factor x or $\bar{x}$ in the top line of identifications reflects the difference in coordinate systems, i.e. real lightcone coordinates versus the exponentiated complex radial plane coordinates. The Casimir shift $-c/24$ has been subtracted from the stress tensor to more closely resemble the stress tensor of [78].

The key point is to identify the central charge of this resulting algebra, comparing the transformation rules of both sides of the identification (3.32) through (3.23) and (2.48) one finds that the central charge is given by

$$c = \frac{3R}{2G}, \tag{3.33}$$

This is the famous Brown-Henneaux central charge [18]. While the central charge emerged as a quantum effect in 2d CFT in asymptotically AdS_3 gravity it already appears at the classical level [69]. This expression also clarifies that if we want our CFT to possess a semi-classical dual, i.e. $R \gg l_{pl}$, that the central charge needs to be very large. There are many alternative ways to arrive at this result, one as alluded to earlier is by computing the Lie algebra of the boundary condition preserving vector fields. Another alternate approach to obtain the Brown-Henneaux central charge is by taking the Einstein-Hilbert action, adding a Gibbons-Hawking term and a particular counter term to make

[2] Within this context the CFT stress tensor $T(x)$ should be thought of as an expectation value instead of an operator, this exact distinction will be covered in the next chapter.

a variational principle well-defined, computing the stress tensor by varying this action with respect to the metric and computing the Weyl anomaly by taking the trace of this stress tensor [75].

3.3 Black hole geometries and their thermodynamics

If empty AdS_3 was all there was to asymptotically AdS 3d gravity interest would be very limited. Fortunately activating the various charges discussed in the previous chapter induces interesting non-trivial solutions. These classes of solutions include black hole solutions which share a lot of the classical and semi-classical quantum properties of the black holes studied by Wheeler, Bekenstein, Hawking and others.

Consider vacuum solutions to Einstein field equations that carry mass. The ADM mass of a geometry is given by the gravitational surface charge generated by the timelike killing vector of the boundary metric at infinite radius [79]. This killing vector field is a constant vector field given by $\zeta_M = \frac{1}{2R}\frac{\partial}{\partial x} + \frac{1}{2R}\frac{\partial}{\partial \bar{x}}$. This vector should be contained within the set of boundary condition preserving vector fields (3.20), indeed it is and it is given by the choices $\xi(x) = \bar{\xi}(\bar{x}) = \frac{R_{AdS}}{2}$. It is insightful to see what this corresponds to in the CFT Virasoro algebra, by making use of the identification (3.32) one finds that the dimensionless mass is given by

$$\begin{aligned}
MR &= -\frac{R}{16\pi G}\left(\int_0^{2\pi} L(x)dx + \int_0^{2\pi} \bar{L}(\bar{x})d\bar{x}\right) \\
&= -\frac{1}{2\pi i}\left(\oint \frac{dz}{z}T(-i\log(z)) + \oint \frac{d\bar{z}}{\bar{z}}\bar{T}(-i\log(\bar{z}))\right) \\
&= -\frac{c}{12} + \frac{1}{2\pi i}\oint dz\, zT(z) + \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}\bar{T}(\bar{z}) \\
&= L_0 + \bar{L}_0 - \frac{c}{12}
\end{aligned} \tag{3.34}$$

comparing with (2.20) ensures that the gravitational sector of AdS gives us the familiar part of the AdS/CFT dictionary that energy on the gravity side corresponds to scaling dimension on the CFT side. An entirely similar analysis can be done for the angular momentum of a geometry, in which case the relevant Killing vector field is a constant vector field of the form $\zeta_l = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial \bar{x}}$. As a result the angular momentum is given by

$$l = L_0 - \bar{L}_0. \tag{3.35}$$

Take this as an opportunity to study the simplest geometries, those whose only non-zero conserved charges are mass and angular momentum. These correspond to stress tensor expectation values on the radial plane that only possess a single double pole at the origin and nothing else, $T(z) = h/z^2$, $\bar{T}(\bar{z}) = h/\bar{z}^2$. In lightcone coordinates this stress tensor is given by $T(x) = h - c/24$, the identification (3.32) tells us that geometries whose only non-zero conserved charges are mass and angular momentum are given by Bañados geometries with constant $L(x)$ and $\bar{L}(\bar{x})$. By applying the Brown-Henneaux central charge (3.33) and the identification (3.32) it can be found that in terms of the CFT stress tensor the Bañados constants are given by

$$L = \frac{4G}{R}h - \frac{1}{4}, \tag{3.36}$$

$$\bar{L} = \frac{4G}{R}\bar{h} - \frac{1}{4}. \tag{3.37}$$

In the original set of t, r, ϕ coordinate and constant values of the parameter functions the Bañados geometries take the form

$$\begin{aligned} ds^2 = &\frac{R^2}{r^2}dr^2 + \left(L + \bar{L} - \frac{r^2}{R^2} - \frac{R^2}{r^2}\right)dt^2 \\ &+ R^2\left(L + \bar{L} - \frac{r^2}{R^2} - \frac{R^2}{r^2}\right)d\phi^2 + 2R(L - \bar{L})dtd\phi \end{aligned} \tag{3.38}$$

Now consider the CFT vacuum stress tensor $T(z) = \bar{T}(\bar{z}) = 0$, this corresponds to the Bañados geometry associated to $L = \bar{L} = -\frac{1}{4}$, performing the coordinate transformation

$$\rho^2 = r^2 - \frac{1}{2}R^2 + \frac{R^4}{16r^2}, \tag{3.39}$$

brings the vacuum Bañados geometry to the form

$$ds^2 = -\frac{\rho^2}{R^2}dt^2 + \frac{R^2}{\rho^2}dr^2 + \rho^2 d\phi^2, \tag{3.40}$$

which can be recognized as the empty AdS metric in Poincaré coordinates. A different choice of coordinate transformation would have brought the Bañados geometry to the full global AdS coordinates [66]. This asserts two things: for one we have the familiar statement that the CFT vacuum state corresponds to empty AdS, secondly inserting the vacuum state into the mass formula (3.34) and applying the Brown-Henneaux formula reproduces the statement that empty AdS$_3$ has a dimensionless mass of $-8R_{AdS}/G$.

Another class of interesting solutions is those corresponding to constant positive values for the parameter functions, $L > 0, \bar{L} > 0$. In which case the coordinate transformation

$$\rho^2 = R^2 \left(L + \bar{L} - \frac{r^2}{R^2} - \frac{R^2}{r^2} \right) \tag{3.41}$$

brings the metric to the form [69]

$$ds^2 = -\frac{1}{R^2\rho^2}(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)dt^2 + \frac{R^2\rho^2 d\rho^2}{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)} + \rho^2 \left(d\phi - \frac{\rho_+\rho_-}{R\rho^2} dt \right)^2, \tag{3.42}$$

where the constants $\rho_\pm$ are defined through

$$L = \frac{1}{4R^2}(\rho_+ + \rho_-)^2, \qquad \bar{L} = \frac{1}{4R^2}(\rho_+ - \rho_-)^2. \tag{3.43}$$

This metric can be recognized as the metric of the BTZ black hole [63], the constants $\rho_\pm$ describe the locations of the inner and outer horizon. These geometries correspond to CFT energy eigenstates with a scaling dimensions greater or equal to $c/24$. All intermediate states with $-1/4 < L, \bar{L} < 0$, or equivalently CFT states with scaling dimensions smaller than $c/24$ correspond to conical spacetimes. These spacetimes were originally dismissed as unphysical in [63], due to the presence of a naked singularity. In later treatments these geometries were included [81] as they show some familiar properties, in particular they can merge together to form BTZ black hole geometries [28]. As such they fit an analogous role within AdS$_3$ gravity to neutron stars, planets or other astrophysical objects without an event horizon. These geometries and their CFT states are summarized in table 3.1.

Another consequence is that the set of BTZ black holes has a mass gap, there is such a thing as a minimal black hole mass which is parametrically separated from the mass of empty AdS. This is a typical feature of black holes in Anti-de sitter space that is not unique to three dimensions[3] [80].

[3]As pointed out by K. Papadodimas this statement requires some clarification, unlike in 3d in higher dimensional AdS spaces less-than-minimal mass black hole solutions do exist, the issue is that these solutions are quantum mechanically unstable and will decay.

Geometry	$L, \bar{L}$ (Bañados metric)	$h, \bar{h}$ (CFT state)
Empty AdS	$-\frac{1}{4}$	0
Conical defect	$-\frac{1}{4} < L, \bar{L} < 0$	$0 < h, \bar{h} < \frac{c}{24}$
BTZ black hole	$L, \bar{L} \geq 0$	$h, \bar{h} \geq 0$

Table 3.1: The three categories of geometries with all Virasoro charges equal to zero except mass and angular momentum, along with their corresponding Bañados parameters and CFT scaling dimensions.

As will be covered extensively in the upcoming chapters, due to the fact that all physical Virasoro coadjoint orbits contain a primary representative this classification exhausts the full set of stationary geometries. That is, up to boundary condition preserving asymptotic diffeomorphisms which can induce boundary gravitons. These boundary gravitons carry energy and have to be taken into account when trying to quantify the thermodynamic properties of a state after equilibriation.

3.3.1 BTZ black hole thermodynamics

The current interest in the study of black holes largely originates from the apparent clash between classical general relativity and quantum mechanics. While typically the regimes of interest of general relativity and quantum mechanics are far apart they clash in the study of black holes, as was famously argued by Hawking [26]. Starting with Hawking's original proof that black hole area never decreases in analogy to the second law of thermodynamics, it was found that all laws of thermodynamics have equivalent manifestations in terms of black hole laws of physics.

The BTZ black holes are a little peculiar in the sense that they are locally everywhere gauge equivalent to AdS$_3$, but they turn out to be no exception to the usual thermodynamic properties of black holes. Just like their higher dimensional AdS-Schwarzschild counterparts they possess a Hawking temperature [7]. A typical shortcut to the Hawking temperature comes from demanding regularity of the near-horizon geometry after Wick rotating to Euclidean time. Take the non-rotating BTZ geometry (3.42), i.e. with $L = \bar{L}$, in this case the inner horizon goes to zero, see (3.43). After Wick rotating $t \to -i\tau$ and taking the limit $\rho \to \rho_+$ and ignoring the angular coordinate the geometry reduces to

$$ds^2 = \frac{2\rho_+}{R^2}(\rho - \rho_+)d\tau^2 + \frac{R^2}{2\rho_+(\rho - \rho_+)}d\rho^2, \tag{3.44}$$

performing the coordinate transformation

$$Q = 2R\sqrt{\frac{\rho - \rho_+}{2\rho_+}}. \tag{3.45}$$

This brings the metric to the one of the Euclidean plane after rescaling the Euclidean time coordinate τ. This resulting Euclidean plane avoids a conical defect exclusively if τ is subject to the identification $\tau \sim \tau + 2\pi\frac{R}{\rho_+}$, finally telling us that the non-rotating BTZ black hole has a temperature given by

$$T = \frac{\rho_+}{2\pi R^2}. \tag{3.46}$$

At this point it is not entirely clear why the correct temperature comes out after demanding that the near-horizon geometry avoids a conical defect while actual conical defect geometries were allowed within the physical spectrum. As such this approach should just be considered as a shortcut to the correct answer which can alternatively be obtained through more involved means [63],

Expression (3.37) lets one express the dimensionless mass in terms of L, leading to

$$MR = \frac{c}{3}L = \frac{1}{8}\frac{\rho_+^2}{GR}, \tag{3.47}$$

this allows one to calculate the black hole entropy

$$\frac{dS}{dM} = \frac{1}{T} \qquad \Longrightarrow \qquad S = \frac{4\pi R\sqrt{M}}{\sqrt{8G}} = \frac{2\pi\rho_+}{4G}. \tag{3.48}$$

The last equality is consistent with the well-known Bekenstein-Hawking formula for the black hole entropy

$$S_{BH} = \frac{A}{4G}, \tag{3.49}$$

where A is the area of the black hole horizon. This simultaneously informs us that the temperature derived above is consistent and that the BTZ black holes obey similar thermodynamic relations as their higher dimensional counterparts. Finally by reintroducing the central charge and eliminating the horizon radius in favor of the temperature the black hole entropy can be recast in the following form

$$S = \frac{2}{3}\pi^2 c \frac{T}{R}, \tag{3.50}$$

this expression has a familiar CFT counterpart.

3.3.2 Modular invariance and the Cardy formula

The central object of study when considering a quantum system coupled to a heat bath is the partition function

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}, \tag{3.51}$$

with inverse temperature $\beta = 1/T$. It is commonly known fact of quantum field theory that the thermal partition function is related to the vacuum partition function of the Euclidean quantum field theory by periodically identifying Euclidean time, the period of identification of Euclidean time being equal to the inverse temperature i.e. $\tau \sim \tau + \beta$.

This has some additional strong implications for the 2d CFTs of the previous chapter. In 2d CFT on the cylinder the spatial direction also possesses a periodic identification $\phi \sim \phi + 2\pi$, hence if we couple 2d CFT to a heat bath then the Euclidean CFT is effectively placed on a flat torus. The flat torus is invariant under the transformations that preserve the ratio of the two independent identifications, this generates the $SL(2, \mathbb{Z})$ discrete group of modular transformations. Demanding that the Euclidean action respects this group of transformations imposes a strong constraint on the partition function of 2d CFT

$$Z(\beta) = Z\left(\frac{4\pi^2}{\beta}\right), \tag{3.52}$$

the power of this expression is that it allows one to derive a universal formula for the high-temperature regime of the partition function. Consider the low-temperature regime $\beta \gg 1$, the expression (3.51) implies that the partition function will be dominated by the CFT vacuum state which has an energy on the cylinder given by $E_0 = -\frac{c}{12R}$ where R is the radius of the cylinder. This means that at low-temperature the partition function can be well approximated by

$$Z(\beta \gg 1) \approx e^{-\beta E_0} = e^{\frac{\beta c}{12R}}. \tag{3.53}$$

Take note that this approximation only works if there exists a gap between the vacuum energy and the energy of the first excited state[4]. Making use of the modular invariance through (3.52) states that at high temperatures the partition function is given by

$$Z(\beta \ll 1) \approx e^{\frac{\pi^2 c}{3\beta R}}. \tag{3.54}$$

The entropy of a quantum system is very closely related to the partition function through

$$S = \frac{d}{dT} T \log(Z(1/T)), \tag{3.55}$$

one can easily see that at high temperatures the entropy of 2d CFT is universal and given by

$$S(T \gg 1) = \frac{2}{3}\pi^2 c \frac{T}{R}. \tag{3.56}$$

This expression has a name and is typically referred to as the Cardy formula, originally derived in [83]. If we identify the radius of the Euclidean cylinder with the AdS radius then this expression matches exactly with the black hole entropy of the BTZ black holes (3.50). Also notice that the high temperature partition function far overshoots the high temperature partition function of a single conformal family, take a look at (2.67) for instance. This tells us that the operator spectrum of a generic CFT needs to contain a sharp growth of operators at large scaling dimensions. This simple identification reveals a lot about the nature holographic CFTs. Three of these things can be summarized thusly:

- The number of high temperature states of a gapped CFT matches the amount of states of BTZ black hole with the same Hawking temperature.
- In order for the Cardy formula to match the Bekenstein-Hawking entropy there needs to be a parametric separation in the operator spectrum between scaling dimensions of the lowest non-trivial primary operator and the identity operator.
- The operator spectrum of the CFT needs to possess a large growth of operators at large scaling dimensions in order to ensure that the CFT spectrum contains enough states to furnish the Cardy formula at high temperatures.

[4]That is, of the CFT hamiltonian, i.e. the first primary in the operator spectrum other than the identity operator needs to have a scaling dimension dimension separated from zero.

All of these conclusions are universal, they did not in any way depend on the action or other microscopic details of the CFT. This is of course the major running theme of this thesis.

3.4 The Chern-Simons formulation of AdS$_3$ gravity

It was emphasized that 3d gravity finds itself in the unusual situation where the theory contains no local degrees of freedom. There exists a well known reformulation of gravity where this feature is made manifest. By appropriate field transformations it can be shown that the Einstein-Hilbert action is equivalent to two copies of a Chern-Simons theory [84, 54]. Chern-Simons theory is the most well-known example of a topological field theory, hence this transformation makes the topological nature of 3d gravity apparent.

The field transformation are constructed from the Palatini formalism of gravity where the metic and Christoffel symbols are treated as independent fields, in which case the relation between the metric and Christoffel symbols manifests itself through the equations of motion [84, 67]. After decomposing the metric into tetrads

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu, \tag{3.57}$$

and the Christoffel symbols into tetrads and the spin connection

$$\Gamma^\sigma_{\mu\nu} = e^\sigma_a \omega^a_{b\nu} e^b_\mu + e^\sigma_a \partial_\nu e^a_\mu. \tag{3.58}$$

The following Lie algebra-valued linear combination of fields can be constructed

$$\mathcal{A}_i = e^a_i P_a + \omega^a_i M_a, \tag{3.59}$$

where P_a are the momentum generators and M_a indicate the Lorentz generators of $SO(2,2)$. The Einstein-Hilbert action is now equivalent to a Chern-Simons theory with gauge field $\mathcal{A}_i$ with gauge group $SO(2,2)$. This is not the most convenient form though, for that purpose the factorization $SO(2,2) = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ can be applied

$$A_i = (\omega^a_i + \frac{1}{R} e^a_i) J^+_a, \quad \bar{A}_i = (\omega^a_i - \frac{1}{R} e^a_i) J^-_a \tag{3.60}$$

where $J^\pm_a$ are the three generators of $SL(2,\mathbb{C})$ in the fundamental representation [85]. In which case the Einstein-Hilbert action turns into

$$S_{EH}(e,\omega) = S_{CS}(A) - S_{CS}(\bar{A}), \tag{3.61}$$

where

$$S_{CS} = \frac{k}{4\pi}\int_M \text{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right). \tag{3.62}$$

The Chern-Simons level k relates back to the AdS radius and Newton's constant through

$$k = \frac{R}{4G}. \tag{3.63}$$

The equations of motion of this action are straightforward, they simply demand that the connection is flat everywhere on the manifold M, or equivalently stated, that the field strength 2-forms vanish everywhere

$$F^a = \bar{F}^a = 0 \tag{3.64}$$

If we take the following basis for the $SL(2,\mathbb{C})$ generators

$$J_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad J_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{3.65}$$

the following property will hold[85]

$$\text{Tr}(J_a J_b) = \delta_{ab}. \tag{3.66}$$

This allows one to invert the gauge connection fields back to a metric field through

$$g_{\mu\nu} = \frac{R^2}{4}\text{Tr}\left((A-\bar{A})(A-\bar{A})\right). \tag{3.67}$$

There is one thing that might potentially require care, the class of solutions of Chern-Simons theory is larger than the class of solutions to the Einstein-Hilbert action. The reason for this is that the resulting gauge field may not necessarily relate back to an invertible metric. An interesting question is what the gauge field manifestation of the Bañados geometries would be. This is in fact known, they correspond to

$$A = b^{-1}(a+d)b, \quad \bar{A} = b(\bar{a}+d)b^{-1}, \quad \text{with} \quad b = e^{\rho J_0}, \tag{3.68}$$

where

$$a = \left(J_1 - \frac{2\pi L}{k}J_{-1}\right)dz, \quad \bar{a} = \left(J_{-1} - \frac{2\pi L}{k}J_1\right)d\bar{z}. \tag{3.69}$$

by substituting this back into (3.67) it can be shown that this indeed reduces to the Bañados metric (3.21).

Chapter 4

Semi-classical CFT and heavy states

While 2d CFTs share a lot of universal features they are not all alike, one parameter that above all else has a large impact on the dynamics and local degrees of freedom of a CFT is the central charge. The central charge c made its first appearance in chapter 2, representations of the conserved charge Dirac algebra form projective representations of the spacetime algebra, the central charge played the role of the generator in the charge algebra that rotated the charge representations in the projective direction.

This original role is very abstract and furthermore suggestive that the actual numerical value of the central charge is largely irrelevant. This turns out to be very much untrue, the central charge has a physical interpretation as counting the number of local degrees of freedom. The most direct way to get an intuition for this statement is to consider a theory of N_b free scalars and N_f free fermions, computing the stress tensor of this free action and computing the two-point function of stress tensors through Wick contraction results in the simple central charge $c = N_b + \frac{1}{2}N_f$.

A more sophisticated general argument in 2d is provided by the c-theorem of A. B. Zamolodchikov [37]. Conformal field theories manifest in the renormalization group flow of quantum field theories as fixed points. The usual intuition behind renormalization is that as we flow down to lower energies, we lower the effective energy cut-off as a result we lose information of high energy degrees of freedom. This intuition was placed on more firm footing in

2d by the c-theorem, which proves that there exists a function of the coupling constants that is monotonously decreasing along the RG flow from high to low energies, and whenever this flow reaches a fixed point the value of this function coincides with the central charge, hence the name c-theorem and the interpretation of c as being proportional to the number of local degrees of freedom.

The first major results in 2d CFT were obtained for central charges that fall within a sequence of rational numbers smaller than one. These rational CFTs have the property that unitarity restricts the number of conformal families to a restrictive finite set of scaling dimensions. The freedom in picking how to contract operators into OPE pairs in higher point functions presents a set of consistency conditions on the OPE coefficients, the conformal bootstrap. The finite nature of the allowed set of conformal dimensions of reduces the bootstrap equations to a set of finite quadratic equations of the OPE coefficents that can be solved analytically. More than just an academic excercise these minimal models turned out to correspond to condensed matter systems at criticality, such as the 2d Ising and Potts model [86] [47].

While these results are very powerful, moving beyond rational central charges turned out to be very challenging. This significant bump in analytical difficulty is sometimes referred to as the $c = 1$ barrier [87, 88]. At the other extreme end of the spectrum there exists another regime though where it is possible to obtain analytical results, this is the regime of large central charge, $c \to \infty$. It is no coincidence that this is exactly the regime where the Brown-Henneaux formula (3.33) states that it is possible for the CFT to have a semi-classical AdS$_3$ gravity dual. In fact it is useful to obtain a sense of scale for the central charge, if we take the AdS radius to be of the order of the Hubble length of our universe, i.e. $10^{26}\,m$ then after reinstating $\hbar$ and c the central charge necessary in order to mimic the scales of our universe is $c = L_H/L_{Pl} \sim 10^{61}$.

The regime of large central charge is usually called the semi-classical regime, the original nomenclature was inspired by Liouville theory where a large central charge corresponds to a large numerical prefactor in the action akin to taking the $\hbar \to 0$ limit in quantum field theory [89]. A nicer post hoc explanation comes from the study of Virasoro coadjoint orbits, which form a natural phase space of theory at large c. The symplectic structure on the Virasoro coadjoint orbits is proportional to the central charge and the scaling dimension of the primary representative of an orbit element. If these are both very large the symplectic form will be very large suggesting that if the system is quantized that the appropriate Planck constant is very small [54].

The large c limit has a similar effect on 2d CFT as the large N ’t Hooft limit in higher dimensional gauge theory. The large N limit induces a factorization of correlators of gauge invariant operators that create low-energy excitations (low in energy compared to N). The large c limit, assuming a sparse spectrum and an assumption mentioned below, has a similar effect. Correlation functions of operators with small scaling dimensions compared to the central charge will factorize into lower-point correlation functions. While this trivializes the dynamics of the low-lying sector of the operator spectrum up to $1/c$ corrections the truly interesting sector of the theory corresponds to operators with scaling of the same order as the central charge. It is the state duals of these operators that can potentially equilibriate into interesting thermodynamic systems. On the other side of the duality recall that the minimal mass BTZ black hole corresponds to a CFT primary state of scaling dimensions $h = \bar{h} = c/24$, it is these heavy states that can potentially have a dual gravitational interpretation in terms of black hole geometries or geometries that will eventually collapse to black holes.

The remaining chapters will cover the dynamics of these heavy CFT states, in the process we will see that there exists a strong interconnection between the dynamics of these states, thermodynamics, AdS$_3$ gravitational physics, and the monodromy properties of linear differential equations with regular singular points.

4.1 Zamolodchikov scaling of heavy correlators

The supposed simplification of 2d CFT correlators in the regime of large central charge largely rests on an assumption for the scaling behavior of conformal blocks of heavy operators. At this point it is important to quantify exactly what it means for an operator to be heavy, an operator is said to be heavy if the ratio of the scaling dimension and the central charge remains finite in the large c limit, i.e.

$$\lim_{c\to\infty} \frac{h}{c} = \text{finite}. \tag{4.1}$$

After decomposing correlation functions of heavy operators into conformal blocks

$$\langle O_1(z_1)...O_n(z_n)\rangle = \sum_h c_{h,h_n}\mathcal{F}_h(h_n, z_n), \tag{4.2}$$

the conformal blocks are assumed to be proportional to

$$\mathcal{F}_h(h_n, z_n) \sim e^{-\frac{c}{6}f_h(h_n,z_n)}. \tag{4.3}$$

The argument as to why this scaling law holds is due to Liouville theory [90]. Conformal blocks are fixed entirely by the Virasoro algebra, hence if this scaling law holds true for the conformal blocks in Liouville theory it should hold true for 2d conformal field theory in general. What makes Liouville theory a particularly useful testing ground is that it contains a continuous spectrum of primary operators, hence one can project out conformal blocks of any desired scaling dimension [89].

Besides the scaling law above, there exists another extremely useful scaling assumption on the behaviour of heavy conformal conformal which is the light operator decoupling from heavy correlators [50]. Take the operators $O_i(z_i)$ to be heavy, if the operator $\phi(z)$ is light in the sense that $\lim_{c\to\infty} h_\phi/c = 0$ then its effect within a correlator is given by

$$\langle \phi(z) O_1(z_1)...O_n(z_n)\rangle \sim g(z, z_i)\langle O_1(z_1)...O_n(z_n)\rangle. \tag{4.4}$$

The main statement is that the function $g(z, z_i)$ is of order $\mathcal{O}\left(c^0\right)$, it is not insenstive to the scaling dimensions of the scaling dimensions of the heavy operators but will generally only depend on these numbers through the ratios h_i/c.

4.1.1 Liouville theory in the semi-classical limit

The action of Liouville theory is given in [91] by[1]

$$S_L = \frac{1}{16\pi b^2}\int d^2x\,\sqrt{g}\left(g^{ab}\partial_a\phi\partial_b\phi + 2bQR\phi + 16\pi b^2\mu e^{\phi}\right), \tag{4.5}$$

where b is a real parameter and R is Ricci scalar of the metric g^{ab} and the constant Q is related to b through $Q = b + b^{-1}$. The Liouville field ϕ is not one of the primary or even one of the descendent fields of the theory, instead the primary fields are exponentials of the Liouville field

$$\text{primary:}\quad e^{\alpha\phi/b}, \qquad \text{scaling dim.:}\quad h = \bar{h} = \frac{1}{2}\alpha(Q-\alpha). \tag{4.6}$$

Finally the central charge of the theory is given by

$$c = 1 + 6Q^2, \tag{4.7}$$

the large central charge limit is taken by taking the $b \to 0$ limit. The goal will be to check what the effect of the $b \to 0$ limit will be on correlators of primary

[1] the form presented here is obtained after performing the field redefinition $\phi \to \frac{\phi}{2b}$.

fields, for this particular purpose we will want to gauge fix the background metric to the flat metric $ds^2 = dzd\bar{z}$, additionally taking the limit $b \to 0$ action reduces to the form [90, 91]

$$S_L = \frac{1}{16\pi b^2}\int_D d^2z\,\left(\partial_a\phi\partial^a\phi + 16\pi\mu b^2 e^{\phi}\right) + \frac{1}{2\pi b^2}\int_{\partial D}\phi\,dx + \frac{2}{b^2}\log(R) + \mathcal{O}(b^2). \tag{4.8}$$

The integration region D corresponds to a disk with large radius R. The boundary terms require some comment, 2d CFT naturally does not live on the flat 2d plane but on the sphere S^2. The sphere is incompatible with the flat metric used to gauge fix the background field though. To get around this issue one can impose a boundary condition at infinity for the Liouville field

$$\phi\left(|z| \to \infty\right) = -2Q\log(z\bar{z}) + \mathcal{O}(1), \tag{4.9}$$

this boundary condition mimics the curvature of the sphere, but also induces the boundary term in the gauge-fixed Liouville action [91].

Take a correlator of primary fields

$$\langle e^{\alpha_1\phi(z_1,\bar{z_1})}...e^{\alpha_n\phi(z_n,\bar{z_n})}\rangle = \int \mathcal{D}\phi\, e^{-S_L} e^{\sum_i^n \alpha_i\phi(z_i,\bar{z}_i)/b} \tag{4.10}$$

In order for the operator insertions to shift the saddle point of the action in the semi-classical limit assume the ratio $\eta_i = \alpha_i/b$ to remain finite and make a similar assumption for the cosmological constant μ:

$$\lim_{b\to 0}\alpha_i b \equiv \eta_i = \text{finite}, \qquad \lim_{b\to 0} b^2\mu = \text{finite}. \tag{4.11}$$

In this limit, the path integral (4.10) is dominated by the saddle points of the shifted action[2]

$$\tilde{S}_L = S_L + \int d^2z \sum_i \frac{\eta_i}{b^2}\phi(z,\bar{z})\delta^2(z - z_i, \bar{z} - \bar{z}_i), \tag{4.12}$$

which are determined by the equation of motion

$$\partial_z\partial_{\bar{z}}\phi = 2\pi\mu b^2 e^{\phi} - 2\pi\sum_i \eta_i\delta^2(z - z_i, \bar{z} - \bar{z}_i). \tag{4.13}$$

[2] In reality the heavy operator insertions need to be regulated as the contributions to the action of these terms diverges. This point will be ignored here, see [91] for details.

All the terms in the equation of motion are of order $\mathcal{O}(b^0)$, if a solution ϕ_s to the equation of motion exists it will be of the same order. Substituting this solution back into the path integral (4.10) hence results in

$$\langle e^{\alpha_1\phi(z_1,\bar{z_1})}...e^{\alpha_n\phi(z_n,\bar{z_n})}\rangle \overset{b\to 0}{=} e^{-\frac{1}{b^2}f(z_i,\bar{z_i},\eta_i)} = e^{-\frac{c}{6}f(z_i,\bar{z_i},\eta_i)}, \tag{4.14}$$

the main assumption of the Zamolodchikov scaling limit is that this scaling behaviour of the full correlator is assumed to carry over to the scaling behaviour of the conformal blocks from which it is constructed. It is instructive to consider what happens if the scaling of the conformal dimension of a primary field is relaxed. Looking back at (4.6), in order for the scaling dimension h_β to be $\mathcal{O}(1)$ take the number β to scale as $\lim_{b\to 0}\beta/b = \nu$ with ν finite. Then adding a operator $\exp(\nu\phi)$ to the correlator will to leading order not effect the saddle point, as a result

$$\langle e^{\nu\phi(z,\bar{z})}e^{\alpha_1\phi(z_1,\bar{z_1})}...e^{\alpha_n\phi(z_n,\bar{z_n})}\rangle \overset{b\to 0}{=} e^{-\frac{c}{6}f(z_i,\bar{z_i},\eta_i)}e^{\nu\phi_s}, \tag{4.15}$$

this is consistent with the decoupling rule (4.4) discussed earlier.

The discussion of Liouville theory so far as acted as a testing ground for the scaling rules of heavy correlators quoted earlier. But there is another way in which Liouville sneaks into the upcoming discussion. A general method for constructing the factorized light operator functions will be based on the uniformization of punctured Riemann surfaces, the metrics of various Riemann surfaces are all related to each other through a local Weyl factor. The Weyl factor that brings a generic metric to surface of constant negative curvature has to satisfy the Liouville equation of motion, with additional delta function sources in case of punctures [92]. This Weyl factor is closely related to the pull-back necessary in order to uniformize the surface, this will be discussed in more detail in section 4.4.

4.2 Monodromy method

While these scaling rules are interesting it is not immediately clear how they can be practically applied to compute physical information from the conformal correlators. The topic of this section and the next will showcase two closely related methods on how this can be achieved, the goal will be to review and in the process compare and contrast these methods.

The first method to be discussed is the one that has been available in the literature for longer and is hence better known, the monodromy method.

This approach ultimately follows from the fact that given the fact that the light operator decoupling applies to all light operators including very special ones.

It was emphasized a few times in chapter 2 that the states at a given level of a generic Virasoro representation are not necessarily linearly independent. Consider this statement in a bit more detail (following [47]), the lowest level at which this could occur is level two, i.e. it is possible that a vanishing linear combination of the two level 2 state vectors can be found

$$L_{-2}|h\rangle + \alpha L_{-1}^2|h\rangle = 0 \tag{4.16}$$

acting on this state with L_1 and L_2 respectively shows that $\alpha = -3(2h+1)/2$ but only if h is related to the central charge through

$$c = 2h\frac{5-8h}{2h+1}. \tag{4.17}$$

Requiring that the the central charge is very large indicates that the scaling dimension of the primary state has to approach $-1/2$, making it an unphysical ghost state. It is in fact a light state though as h approaches closer to $-1/2$ as c grows implies that $\lim_{c\to\infty} h/c$ vanishes, the point is merely to exploit the lightness of this unphysical state to gain constraints on the physical correlator that remains after decoupling the ghost state.

Consider the operator dual of the state $|h\rangle$, this is a primary operator $\tilde{\psi}(z)$ with a scaling dimension matching that of the ghost state. The consideration above suggest that it is eliminated by the combination

$$[L_{-2}, \tilde{\psi}(z)] + \frac{c}{6}[L_{-1}, [L_{-1}, \tilde{\psi}(z)]] = 0, \tag{4.18}$$

here the scaling dimension h has been eliminated from the constant α in favor of the central charge. Let us now check how this lends itself to the problem of calculating the correlator

$$\langle O_1(z_1)...O_n(z_n)\rangle, \tag{4.19}$$

where all the operators O_i are heavy. Insert the operator $\tilde{\psi}(z)$ into correlator above, from the light operator decoupling formula (4.4) this results in

$$\langle\tilde{\psi}(z)O_1(z_1)...O_n(z_n)\rangle = \psi(z)e^{-\frac{c}{6}f(z_i)}, \tag{4.20}$$

imposing the shortening condition (4.18) constrains the decoupled function

$$\begin{aligned}
0 &= \langle [L_{-2}, \tilde{\psi}(z)] + \frac{c}{6}[L_{-1}, [L_{-1}, \tilde{\psi}(z)]] O_1(z_1)...O_n(z_n)\rangle \\
&= \frac{c}{6}\partial_z^2 \langle \tilde{\psi}(z) O_1(z_1)...O_n(z_n)\rangle + \frac{1}{2\pi i}\langle \oint_z dz' \, \frac{T(z')}{z-z'}\tilde{\psi}(z) O_1(z_1)...O_n(z_n)\rangle \\
&= \frac{c}{6}\psi''(z)\langle O_1(z_1)...O_n(z_n)\rangle + \psi(z)\langle T(z) O_1(z_1)...O_n(z_n)\rangle. \qquad (4.21)
\end{aligned}$$

Let us now define the normalized stress tensor expectation value $T(z)$ through

$$T(z) \equiv \frac{\langle T(z) O_1(z_1)...O_n(z_n)\rangle}{\langle O_1(z_1)...O_n(z_n)\rangle}, \qquad (4.22)$$

this is not quite an expectation value yet, but it will be in all applications in upcoming sections so it useful to introduce the name in advance. Including this definition the the condition (4.21) reduces to the form of the linear ODE

$$\psi''(z) + \frac{6}{c}T(z)\psi(z) = 0. \qquad (4.23)$$

This ODE encodes in principle all the dynamical information contained within the conformal blocks of interest.

4.2.1 stress tensor expectation value

The ODE derived above is parametrized by a function, the stress tensor expectation value. This function is not entirely arbitrary it is in large part fixed by the Virasoro Ward identity. It is useful to study its general form in a bit more detail as it will reveal what sort of ODE (4.23) really is.

The irregular parts of the OPE between the stress tensor and primary operators are fixed by (2.40), applying this expression to the unnormalized stress tensor expectation value results in

$$\langle T(z) O_1(z_1)...O_n(z_n)\rangle = \left(\sum_i^n \frac{h_i}{(z-z_i)^2} + \frac{\partial_{z_i}}{z-z_i}\right)\langle O_1(z_1)...O_n(z_n)\rangle. \qquad (4.24)$$

Exploiting the scaling of heavy correlators

$$\langle O_1(z_1)...O_n(z_n)\rangle = e^{-\frac{c}{6}f(z_i)}, \qquad (4.25)$$

suggests that the heavy correlators factors out of the derivatives in the simple poles resulting in the following expression for the normalized stress tensor expectation value

$$T(z) = \sum_{i}^{n} \frac{h_i}{(z-z_i)^2} + \frac{c_i}{z-z_i}, \tag{4.26}$$

where the accessory paramaters c_i are defined through

$$c_i \equiv -\frac{c}{6}\partial_{z_i} f(z_i). \tag{4.27}$$

Because of the form of $T(z)$ the ODE (4.23) is a differential equation of Fuchsian class with regular singular points at the locations of operator insertions. The strength of the singularities is determined by the scaling dimension of the operators in the correlator.

Further constraints can be imposed on the accessory parameters by considering the asymptotics of $T(z)$ at large z. The stress tensor $T(z)$ decays as z^{-4} at infinity, hence expanding the sum in the expression above around large z, or equivalently small z_i will give us constraints on the accessory parameters

$$\begin{aligned} \sum_{i=1} \frac{c_i}{z-z_i} + \frac{h_i}{(z-z_i)^2} &= \left(\sum_{i=1} \frac{c_i}{z} + \frac{h_i}{z^2}\right) + \left(\sum_{i=1} \frac{c_i}{z^2} + \frac{2h_i}{z^3}\right) z_i \\ &+ \frac{1}{2}\left(\sum_{i=1} \frac{2c_i}{z^3} + 6\frac{h_i}{z^4}\right) z_i^2 + \mathcal{O}(z^{-4}). \end{aligned} \tag{4.28}$$

From this we can read of the constraints:

$$\begin{aligned} &\sum_{i=1} c_i = 0, \\ &\sum_{i=1} h_i + c_i z_i, \\ &\sum_{i=1} 2h_i z_i + c_i z_i^2. \end{aligned} \tag{4.29}$$

These constraints, not coincidentally, pull double duty as the constraints required in order to avoid that the ODE (4.23) has an (ir)regular singular point at infinity.

4.2.2 monodromy constraints

The remaining free accessory parameters hold the key to the conformal blocks, the accessory parameters are given by the derivatives of the exponentiated correlators (4.27). If one could solve for the accessory parameters then the conformal block would follow by means of integration of these parameters.

The solutions to the differential equation (4.23) will generally be multi-valued functions with branch points at the regular singular points induced by the operators insertions. The trick to constraining the accessory parameters is by restricting the monodromies of the solutions. Specifying a conformal block involves specifying how to contract al the operators in OPE pairs and specifying what conformal family is exchanged between them see figure 4.1.

Consider the monodromy along a cycle that encircles two operators that are contracted through the OPE 4.1. Specifying the conformal blocks entails specifying the conformal family to which the two operators are fused, this forces the monodromy feel the presence of the fused operator. This restricts the monodromy matrix around the contour, in a particular basis, to be given by

$$M = \begin{pmatrix} e^{\pi i\sqrt{1-24h/c}} & 0 \\ 0 & e^{-\pi i\sqrt{1-24h/c}} \end{pmatrix}, \tag{4.30}$$

where h is the scaling dimension of the exchanged conformal block. The monodromy matrix is basis dependent, hence it is more advantageous to isolate a property that is basis independent, an example is given by the trace

$$\text{Tr}(M) = -2\cos(\pi\sqrt{1-24h/c}). \tag{4.31}$$

Computing the monodromy along the cycle while being agnostic of the conformal block would give one in principle a monodromy matrix as a function of scaling weights and accessory parameters. Demanding that the traces of these monodromy matrices match provides a set of constraints for the accessory parameters. Solving these contraints and subsequently integrating the set of accessory parameters yields the conformal block. To summarize the algorithm

- Specify a conformal block by choosing an OPE channel and fixing the intermediate conformal families, demand that the monodromy cycle around a contracted OPE pair is consistent with the exchanged operator family.

- Compute the trace of the monodromy matrix of the solutions of (4.23) around a cycle containing the regular singular points corresponding to the OPE pair.

- Demand that both traces match and derive from that a constraint for the accessory parameter.

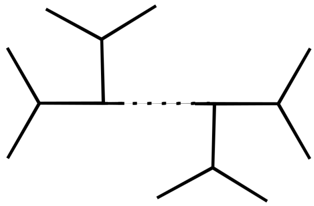


Figure 4.1: Trivalent graph representation of an OPE channel

- Repeat the previous steps with other OPE pairs until enough constraints are obtained such that when combined with the regularity conditions (4.29) all accessory parameters can be solved.

- Integrate all the accessory parameters to obtain the exponentiated conformal block.

While this algorithm is in general extremely difficult to perform analytically, the fact that the problem of computing the conformal blocks can be reforged into a monodromy problem can be a great help conceptually. A large open problem of the AdS/CFT correspondence is to truly flesh out the dictionary of the correspondence. It is often of great conceptual help to match objects and dynamics on both sides of the duality by showing that ultimately both objects correspond to solving the exact same mathematical problem, even if actually solving the resulting problem analytically is intractable.

4.3 The uniformization approach

Another more recent approach to constructing conformal blocks is centered around the classic Poincaré uniformization theorem. To decipher the gravitational physics out of CFT correlators it is very useful to study how light probe operators respond to the presence of heavy operators. One example we already observed was in the form of the light operator decoupling (4.4). The topic of this section will be to assert the result originally obtained in [94] within the context of heavy primary states. Under the assumption that the heavy correlator is dominated by the identity block, then inserting two probe operators on a heavy correlator causes them to behave as if they propagate on top of an effective background geometry.

The assumption that a correlator of heavy operators is given by a single conformal block can be written as

$$\langle O_1(x_1)...O_n(x_n)\rangle \sim e^{-\frac{c}{6}f(x_i,h_i)}, \tag{4.32}$$

the light operator decoupling suggests that the related correlator is given by

$$\langle Q(z_1)Q(z_2)O_1(x_1)...O_n(x_n)\rangle \sim Q(z_i,x_i)e^{-\frac{c}{6}f}. \tag{4.33}$$

For both of these cases the effect of inserting a stress tensor in the correlator is fixed by the operator product expansion between the stress tensor and primary operators

$$\langle T(z)O...O\rangle = \left(\sum_i \frac{h_i}{(z-x_i)^2} - \frac{\frac{c}{6}\partial_i f}{z-x_i}\right) e^{-\frac{c}{6}f} \equiv T(z)e^{-\frac{c}{6}f}, \tag{4.34}$$

note that this expectation value is $\mathcal{O}(c)$. Similarly

$$\langle T(z)QQO...O\rangle = \left(\sum_Q \frac{h_Q}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial_{z_i}Q}{Q}\right) Qe^{-\frac{c}{6}f}$$
$$+\left(\sum_H \frac{H_i}{(z-x_i)^2} - \frac{\frac{c}{6}\partial_i f}{z-x_i} + \frac{1}{z-x_i}\frac{\partial_{x_i}Q}{Q}\right) Qe^{-\frac{c}{6}f} \equiv T_Q(z)Qe^{-\frac{c}{6}f}, \tag{4.35}$$

which is also of order $\mathcal{O}(c)$ but only due to the presence of the terms that also occur in $T_O(z)$. Define the conformal transformation $u(z)$ by the following property

$$\left(\frac{dz}{du}\right)^2 T(z(u)) + \frac{c}{12}S[z,u] = 0, \tag{4.36}$$

this conformal frame is useful as it exploits inhomogeneous nature of the stress tensor transformation rule to subtract out the $\mathcal{O}(c)$ terms from $T_Q(z)$. This procedure is not entirely well-defined though as we will see the function $u(z)$ will contain branch points, which means that $u(z)$ does not admit a Laurent expansion everywhere, which technically disqualifies it as a conformal transformation. We will come back to some of subtleties related to this in chapters 5 and 6. In the meantime performing this transformation leads to

$$\langle T(u)QQO...O\rangle =$$
$$J_Q\left(\frac{dz}{du}\right)^2 \left(\sum_{Q,H} \frac{h_Q}{(z(u)-z_i)^2} + \frac{1}{z(u)-z_i}\frac{\partial_{z_i}Q}{Q} + \frac{1}{z(u)-x_i}\frac{\partial_{x_i}Q}{Q}\right) Qe^{-\frac{c}{6}f}, \tag{4.37}$$

where all the Jacobian factors of the heavy primaries and probes have been collected into the factor J_Q. Notice how the remaining stress tensor only has two remaining double poles at the locations of the probe operators, this raises the suspicion that the background geometry has screened of the remaining operators. This suspicion can be shown to hold by considering the effect of inserting partial projection operators to project the desired conformal block from the original correlator is u-coordinates. We can project out the identity block contribution through

$$\langle QQ\, P_0\, O...O\rangle_u, \tag{4.38}$$

where the matrix P_0 is given as in (2.87) through

$$P_0 = \sum_{\{n_k\},\{m_k\}} F^{-1}(\{n_k\},\{m_k\},0)L_{-n_k}...L_{-n_1}|0\rangle\,\langle 0|L_{m_k}...L_{m_1}. \tag{4.39}$$

The powers of c can be conveniently counted by replacing the Virasoro generators with the inverse Fourier transform of the stress tensor and contracting by means of the TT-OPE. Due to the choice of conformal frame one of the powers of h_H is subtracted out from the numerator terms, as a result the denominators will always be of a greater order in c, the full argument can be found in [94, 95]. The only Virasoro generator that evades this argument is L_{-1}. This can be seen from the Virasoro algebra, passing L_{-1} through L_1 will not yield the central extension term hence the diagonal components of the matrix components F^{-1} will not pick up a power of c. Conveniently this generator annihilates the vacuum state so these terms will not contribute to the sum, hence in the limit $c\to\infty$ the only surviving term is the one associated to the vacuum, i.e.

$$P_0 \overset{c\to\infty}{\to} |0\rangle\langle 0|, \tag{4.40}$$

hence in the uniformizing coordinate frame the identity block contribution of a *heavy* correlator inherits the familiar property that it computes the factorized contribution to the correlator in the large N limit. By transforming back to the original z-coordinates and dividing out all the factors that are independent of z_1 and z_2 one ends up with

$$Q(z_1,z_2) = \left(\frac{du}{dz_1}\right)^{h_Q}\left(\frac{du}{dz_2}\right)^{h_Q}\frac{1}{(u(z_1)-u(z_2))^{2h_Q}}. \tag{4.41}$$

Hence once we go to the appropriate conformal frame computing the leading order contribution in $1/c$ is easy, the catch being that finding the right conformal frame is extremely difficult. In the preceding discussion of Liouville theory it was mentioned that the the curvature of the sphere could be absorbed in the insertion of a background charge operator. This approach is essentially the opposite, the presence of the heavy operators is screened off by an appropriate background geometry. Note that in principle this method also suggests how one could systematically construct the subleading $1/c$ corrections [95].

4.3.1 Uniformization and the Fuchs equation

Often the form presented in the last section is not the most convenient form to express the physics of the probe operators, due to the fact that the Schwarzian differential equation (4.36) is highly non-linear. Right off the bat, it can already be cast into a more familiar form by inversion

$$S[u,z] = \frac{12}{c} T(z), \tag{4.42}$$

this equation is ubiquitous within the literature on uniformization of punctured Riemann surfaces where $u(z)$ takes the form of a ratio of a basis of solutions $\psi_1(z)/\psi_2(z)$ of the the linear second order ODE

$$\psi''(z) + \frac{6}{c} T(z)\psi(z) = 0, \tag{4.43}$$

under the assumption of the following Wronskian normalization condition on the solutions

$$\psi_1'(z)\psi_2(z) - \psi_1(z)\psi_2'(z) = 1. \tag{4.44}$$

The differential equation (4.43) is the exact same equation as the null-vector decoupling equation of the monodromy method. In the case that the operators in the heavy correlator correspond to conformal primaries then once again (4.43) corresponds to an equation of Fuchsian class with regular singular points at the operator insertions. The exact derivation of the Fuchs equation from the Schwarzian equation is presented in appendix B. This derivation also shows that the relation between $\psi(z)$ and $u(z)$ can be inverted to

$$\psi(z) = \frac{1}{\sqrt{u'(z)}}, \tag{4.45}$$

in the sense that if $u(z)$ solves the Schwarzian equation then one can construct a $\psi(z)$ that solves the Fuchs equation. The Schwarzian derivative is invariant under conjunction with a fractional linear transformation, hence both the Schwarzian and Fuchs equation possess the same $SL(2,\mathbb{C})$ freedom in picking a solution. In terms of the Schwarzian equation it corresponds to the freedom of conjoining a solution with a fractional linear transformation and for the Fuchs equation it corresponds to picking a basis in which to express the solution.

It is useful to construct the form the probe correlator takes in terms of solutions to the Fuchs equation, substituting (4.45) into the correlator gives

$$Q(z_1, z_2) = \psi(z_1)^{-2h_Q}\psi(z_2)^{-2h_Q}\left(\int_{z_1}^{z_2} dz\,\psi(z)^{-2}\right)^{-2h_Q}. \tag{4.46}$$

The contour is not arbitrary due to the multi-valuedness of the function $\psi(z)$, but in chapter 5 we will see that there exists a natural contour to take to consider probes that are shifted in time but not space.

4.4 Gravity, uniformization and the Liouville equation

There exists historic context in which both the Schwarzian equation and the Fuchs equation have made focal appearances, this is within the context of the uniformization theorem of punctured Riemann surfaces due to Poincaré [96] and others (for instance, see [92] and references therein). Part of the uniformization theorem states that given a punctured Riemann surface there exists a local Weyl transformation that brings the metric to that of the Poincaré disk, the filled unit disk with constant negative curvature. The appearance of the term negative curvature immediately evokes the notion of anti-de Sitter space and these things are indeed not unrelated as this section will cover.

To probe the dynamics of heavy states it is beneficial to compute probe expectation values on top of heavy out-of-equilibrium states. For this purpose throughout the rest of this thesis we will always consider correlators on the radial plane of the form

$$\langle V|Q(z_1)Q(z_2)|V\rangle \qquad \text{with} \qquad |V\rangle = O_1(x_1)...O_n(x_n)|0\rangle, \tag{4.47}$$

by the manner in which taking Hermitian adjoints works on the radial plane this means that

$$\langle V| = (x_1^*)^{-2H_1}\ldots(x_1^*)^{-2H_n}\,\langle 0|\,O_1(1/x_1^*)...O_n(1/x_n^*), \tag{4.48}$$

though the prefactors will typically be ignored due their independence of the probe operators, the important part is location of the adjoint operators. These

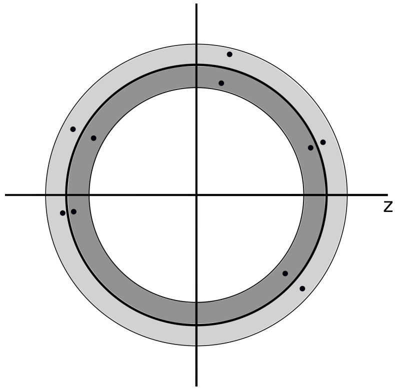


Figure 4.2: A graphical representation of the annuli in which the heavy operators and their adjoints fall. A set of point is given that gives an example of the kind of heavy primary distribution that are under consideration.

are located at insertion points that are reflected around the unit circle with respect to their ket-vector counterparts. This will be covered in much more detail in chapters 5 and 6.

The locations at which the heavy operators are inserted is not quite arbitrary if one wants to ensure that there exists an OPE channel where the identity block dominates. To ensure this the operators are chosen to lie within a thin annulus, the annulus is shown in figure (4.2) along with an example of a state that falls within the class under consideration. The desired OPE channel is then given by the one obtained by contracting the heavy operators and their adjoints into OPE pairs, following [97] these OPE pairs will sometimes be referred to as mirror pairs.

As a secondary notion this implies that the stress tensor expectation value inherits a property under complex conjugation from the hermitean adjoint of the stress tensor operator

$$T(z) = \frac{1}{z^4} T(1/z^*)^* \qquad \text{with} \qquad T(z) = \frac{\langle V|T(z)|V\rangle}{\langle V|V\rangle}. \tag{4.49}$$

It was implicitly observed in [97] that this reflection property ensures that solving the monodromy problem detailed in section 4.2 in order to obtain the identity block is *equivalent* to the monodromy problem associated with the uniformization of the punctured Poincaré disk.

4.4.1 Uniformization of punctured Riemann surface

In the last few sections a lot of reference has been made to Poincaré's classical uniformization theorem, hence it is appropriate to quickly review some of its most famous results. The theorem states that any punctured Riemann surface with genus, $g > 1$ has a universal cover given by the unit disk (or equivalently the upper half-plane). In other words, this theorem proves the existence of a conformal transformation, $w(z) \to z$ that acts as a pull-back from the punctured disk to the constant curvature Poincaré disk. Furthermore, the stress-energy tensor makes a natural appearance here when we cast the problem in terms of Liouville field theory.

This interpretation of the theorem can be rephrased in a particular Riemann surface, namely the punctured disk with n punctures ($\mathbb{CP}^1/\{\zeta_i\}$) (as opposed to CFT the problem on the radial plane with 2n regular singular points) corresponding to the insertion points, ζ_i of our CFT operators inside the unit circle. The uniformization problem in this setup is to find the local Weyl factor $\phi(\zeta, \bar{\zeta})$ in order for the metric $ds^2 = e^{\phi} d\zeta d\bar{\zeta}$ to have a constant negative curvature on the entire punctured disk. The appropriate Weyl factor possesses the desired property on the punctured sphere if it satisfies the Liouville equation [92]

$$\partial_{\zeta}\partial_{\bar{\zeta}}\phi(\zeta, \bar{\zeta}) - \frac{1}{2} e^{\phi(\zeta, \bar{\zeta})} = 0, \tag{4.50}$$

subject to the holomorphic boundary conditions

$$\phi(\zeta) = \begin{cases} -2\log|\zeta - \zeta_i| - 2\log|\log|\zeta - \zeta_i|| + O(1) & \zeta \to \zeta_i \\ -2\log|\zeta| - 2\log|\log|\zeta|| + O(1) & |\zeta| \to 1 \end{cases} \tag{4.51}$$

and similar conditions for the anti-holomorphic part. The first line of boundary conditions ensures that the metric is complete on the disk, the second condition makes sure that the punctured disk under consideration gets mapped to the Poincaré disk of area of $4\pi\,(N-1)$ [92]. A holomorphic solution to the Liouville equation can be related to the holomorphic stress-energy tensor, $T(\zeta)$ by means of[3]

$$T(\zeta) = -\partial_{\zeta}^2\phi(\zeta) - (\partial_{\zeta}\phi(\zeta))^2 \,. \tag{4.52}$$

Similar expression also holds between their anti-holomorphic counterparts.

The only divergent point that $\phi(\zeta)$ is allowed to possess are near the punctures, since the boundary conditions restrict the divergences to be logarithmic it means that $T(\zeta)$ can only have at most second-order poles. Ignoring

[3]note that there exists a sign convention difference between [97] and [92].

its unimportant regular part, the most general form of $T(\zeta)$ as a function of coordinates on the unit disk is given by

$$T(\zeta) = \sum_{i=1}^{N} \frac{\epsilon_i}{(\zeta - \zeta_i)^2} + \frac{c_i}{\zeta - \zeta_i}, \tag{4.53}$$

since the parameters ϵ_i control the strength of the singularities this whole expression is very similar to what appears as the stress-energy tensor expectation value, $T(z)$ in the CFT problem, as long as the state $|V\rangle$ is constructed entirely out of products of primary operators. This connects the uniformization problem we discussed in the text to the classical uniformization problem on the N-punctured Riemann surface. Therefore, the properties of the Liouville stress-energy tensor, $T(\zeta)$ can be attributed to the properties of our stress-energy expectation value, $T(z)$.

The general solution to the Liouville equation is in fact known, by simple substitution it can be shown that

$$\phi(\zeta) = \log\left(\frac{4|f'(\zeta)|^2}{(1-|f|^2)^2}\right) = \log\left(\frac{4f'(\zeta)\bar{f}'(\bar{\zeta})}{(1-f(\zeta)\bar{f}(\bar{\zeta}))^2}\right) \tag{4.54}$$

solves the equation of motion, here $f(z)$ is generic meromorphic function. Hence the difficult part of the uniformization problem is not to find the general solution to the differential equation but to select the appropriate particular solution that solves the boundary condition. In fact by taking $f(z) = z$ you get the Weyl factor which corresponds to a metric of the unit disk with constant negative curvature[98] (i.e. this is the trivial conformal map that maps the complex plane to the Poincaré disk). In order for the metric to be single-valued on the entire disk we need that the fundamental group of $f(z)$ on the disk to be a mapping of the form

$$M:\, \pi_1(\mathbb{CP}^1/\{\zeta_i\}) \to SU(1,1), \tag{4.55}$$

the restriction to $SU(1,1)$ as opposed to $SL(2,\mathbb{C})$ ensures the single-valuedness of the metric on the punctured disk. Furthermore by substituting (4.54) into (4.52) one can demonstrate that $f(z)$ satisfies the property

$$S[f,\zeta] = 2T(\zeta), \tag{4.56}$$

The real boundary condition of the function $f(\zeta)$ on the boundary of the unit disk invokes a reflection property of the resulting stress-energy tensor.

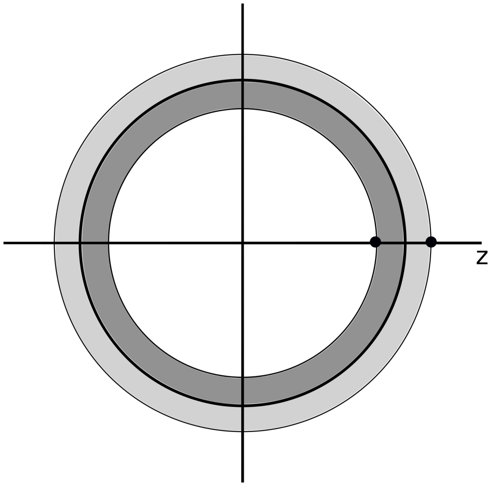


Figure 4.3: A single mirror pair, this set up has the bulk dual interpretation of a a displaced conical defect or black hole.

4.4.2 Primary operators as particles at rest

States created purely by heavy operators appear to play a special role in the previous section, this observation can be expanded on the by the following qualitative picture. One can derive some intuition for primary operators by considering a state generated by a single mirror pair, in this case the stress tensor expectation value is entirely fixed by conformal invariance and the Fuchs equation can be solved exactly. If

$$|V\rangle = O(1-\sigma)|0\rangle, \quad \text{with} \quad \sigma \ll 1 \tag{4.57}$$

then

$$T(z) = \frac{h}{(z-1+\sigma)^2} + \frac{h/\sigma}{z-1+\sigma} + \frac{h}{(z-1-\sigma)^2} - \frac{h/\sigma}{z-1-\sigma}. \tag{4.58}$$

It is known that a state constructed out of a primary operator inserted at the origin has the bulk interpretation of either a black hole or conical defect at rest in the center of AdS_3 [94]. The fractional linear transformations of the CFT correspond to isometries of AdS_3, hence the displaced mirror pair has an interpretation as a displaced, bouncing or sloshing [99] conical defect/black hole, see figure (4.4).

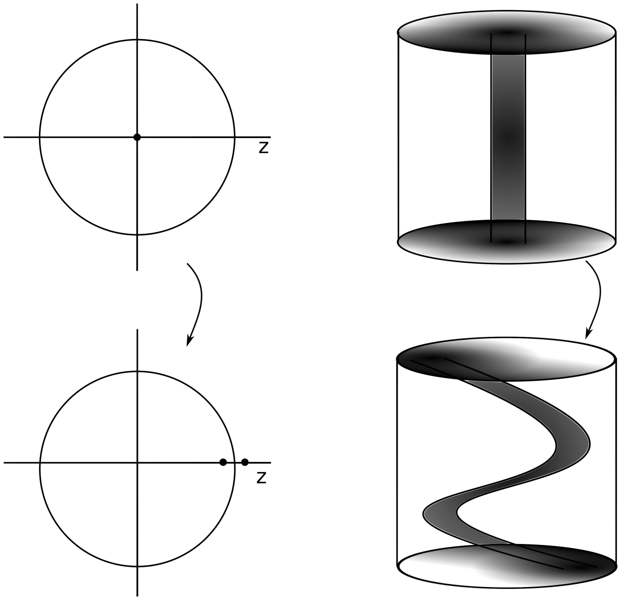


Figure 4.4: The process of displacing a primary operator at the origin from the perspective of the CFT and the bulk dual.

The energy of the state can computed by the appropriate contour integral over the stress tensor. The energy has to computed after the state is created but before it is annihilated again, hence an appropriate choice is to integrate along the unit circle

$$L_0 = \oint_{|z|=1} dz\, zT(z) = \frac{h}{\sigma}. \tag{4.59}$$

By equating this with the integral of motion of a particle initially at rest in global AdS, i.e.

$$E = \sqrt{1 + \frac{r^2}{R^2}} \tag{4.60}$$

and applying the mass formula (3.34) one can obtain the following relationship between $\sigma \ll 1$ and the bulk radial displacement r

$$\sigma \propto \frac{R}{r}. \tag{4.61}$$

This essentially states that small σ corresponds to large displacement of the conical defect towards the AdS boundary, in turn reinforcing the notion that greater identity block dominance corresponds to a regime where the gravitational interaction dominates.

This suggests an alternative interpretation of the unit disk as corresponding to a bulk spatial time-slice in some appropriate AdS coordinates, similar to the picture presented in [97]. This would elucidate the link between primary operators and uniformization. The existence of a coordinate transformation that brings the punctured disk to the Poincaré disk tells us that the original time-slice had constant negative curvature, which is the initial-slice condition of slice with zero extrinsic curvature. This paints the following physical picture, *the insertion of mirror pairs of primary operators on the radial plane near the unit circle has a bulk dual geometry in terms of massive particles that are close to the boundary and initially at rest*.

This analysis is of course very incomplete as the presence of multiple mirror pairs severely complicates the necessary calculations to the point of making them intractable. But the simple one mirror pair set-up is very suggestive.

4.5 Uniformization and Thermodynamics

As was already immediately established in [94] one of the great advantages of studying uniformization, or heavy CFT correlators in general, is that it formalizes a lot of the intuition of statistical mechanics applied to conformal field theory. Particularly [94] notes that heavy primary states with scaling dimensions $h > c/24$ act as typical states of a thermal ensemble of temperature

$$T = \frac{1}{2\pi}\sqrt{24h/c - 1}. \tag{4.62}$$

This can be seen from the fact that for the case of a primary state the appropriate Fuchs equation can be solved analytically. Specifically the differential equation

$$\psi''(z) + \frac{6}{c}\frac{h}{z^2}\psi(z) = 0, \tag{4.63}$$

is simply solved and a basis of solutions is given by

$$\psi_\pm(z) = z^{\frac{1}{2}\pm\frac{1}{2}\sqrt{1-24h/c}}. \tag{4.64}$$

By means of the expression (4.45) this can be expressed as the conformal transformation

$$u(z) = z^{\sqrt{1-24h/c}}. \tag{4.65}$$

In the case that $h > c/24$ this conformal frame periodically identifies Euclidean time through

$$\tau \sim \tau + \frac{2\pi}{\sqrt{1-24h/c}}, \tag{4.66}$$

leading to the temperature quoted above.

The uniformization approach unveils another property of thermal ensembles. Given a typical state of a thermal ensemble $|h\rangle$ one can construct a set of non-equilibrium states that are indistinguishable from the typical state from the perspective of simple operators

$$|h\rangle \longleftrightarrow e^{-\beta H/2}U(0)e^{\beta H/2}|h\rangle, \tag{4.67}$$

where β is the usual inverse temperature $\beta = 1/T$ and H is the Hamiltonian. These states are indistinguishable from each other in the following way, the statement that $|h\rangle$ is a typical state of the thermal ensemble implies that

$$\langle h|Q...Q|h\rangle = \mathrm{Tr}\left(e^{-\beta H}Q...Q\right) + \mathcal{O}\left(e^{-S/2}\right), \tag{4.68}$$

under the replacement (4.67) this changes into

$$\begin{aligned}
&\langle h|\, e^{\beta H/2}U^\dagger(0)e^{-\beta H/2}\; Q...Q\; e^{-\beta H/2}U(0)e^{\beta H/2}\,|h\rangle \\
&= \mathrm{Tr}\left(e^{-\beta H}\; e^{\beta H/2}U^\dagger(0)e^{-\beta H/2}\; Q...Q\; e^{-\beta H/2}U(0)e^{\beta H/2}\right) \\
&= \mathrm{Tr}\left(e^{-\beta H}Q...Q\right) \\
&= \langle h|Q...Q|h\rangle.
\end{aligned} \tag{4.69}$$

All the $\mathcal{O}\left(e^{-S/2}\right)$ error terms have been suppressed. The key step between line and 2 and 3 is due to the exploitation of the cyclicity of the trace [100]. These non-equilibrium states, being indistinguishable from pure states, have recently been examined as possible holographic realizations of behind-the-horizon fluctuations [101].

Interestingly by applying the uniformization approach to constructing conformal blocks discussed in the previous sections the same conclusion can be reached without invoking any thermal ensembles, exposing once again the parallels between leading-order high-energy state physics and thermodynamics. Consider the correlator

$$\langle O_h(0)\, e^{\beta H/2} U^\dagger(0) e^{-\beta H/2}\, Q...Q e^{-\beta H/2} U(0) e^{\beta H/2}\, O_h(\infty)\rangle, \tag{4.70}$$

First note that the transformation $e^{-\beta H/2}$ acts as a shift by $\beta/2$ in Euclidean time. On the radial plane $z = e^{\tau + i\phi}$, hence $\tau \to \tau + \beta/2$ leads to $z \to e^{\beta/2} z$. Our correlator can therefore equivalently be written as

$$\langle O_H(\infty) U^\dagger(e^{-\beta/2}) Q...Q U(e^{\beta/2}) O_H(0)\rangle. \tag{4.71}$$

This correlator is Euclidean and none of the operators are (assumed to be) located at the same point, hence all operators commute with one another and it can be rewritten into the more suggestive form

$$\langle O_H(\infty) O_H(0) U^\dagger(e^{-\beta/2}) U({}^{\beta/2}) Q...Q\rangle. \tag{4.72}$$

At this point we can use the uniformization argument. this correlator has the same heavy sector as above, i.e. consisting of a single heavy primary at the origin and its adjoint. Therefore the same conformal transformation (4.65) remains appropriate. We have to match the scaling dimension h in such a way that it is a typical state of a thermal state with inverse temperature β, for this purpose one can use (4.62) to eliminate h in favour of β. The usual coordinate transformation is $u = z^{\sqrt{1-24H/c}}$, combining these expressions lets us eliminate h and write the relevant uniformizing coordinate as $u = z^{\frac{2\pi}{i\beta}}$. After performing the coordinate transformation the correlator will (to leading order in $1/c$) factorize

$$J(e^{\beta/2}, e^{\beta/2}, x_i) \langle O_H O_H\rangle \langle U^\dagger(e^{i\pi}) U(e^{-i\pi}) Q...Q\rangle, \tag{4.73}$$

where the various Jacobian factors have been absorbed into J. Note the following main point: the coordinate transformation has had the effect of transporting the unitary operators on top of each other, where by their unitary nature they cancel each other. After stripping off all multiplicative factors the result is

$$J(x_i)\langle Q...Q\rangle. \tag{4.74}$$

This is of course exactly what would have been obtained if you had used the same technique to evaluate $\langle O(\infty)Q...QO(0)\rangle$ instead.

Chapter 5

Thermalization in semi-classical CFT

Semi-classical conformal field theories find themselves at an interesting intersection point in the space of physical models. On the one hand they can describe stongly interacting systems in which case they can potentially demonstrate extremely complex collective behaviour such as thermalization and various phase transitions. One the other hand, as was discussed at length in the previous chapter, they are highly constrained by the Virasoro algebra. And they can be even further constrained by additional dynamical assumptions such as a gap in the operator spectrum and the Zamolodchikov scaling discussed previously.

In further addition if we assume these strongly-coupled semi-classical CFTs possess a holographic dual we can derive a lot of intuition for this complicated collective behaviour in terms of semi-classical 3d gravity. The main topic of this chapter will be the process of thermalization, i.e. the process whereby a system starting in an atypical pure state settles down into a subsector of typical pure states of an appropriate thermal ensemble. Being inherently the result of large amounts of interactions within a huge many-body system, thermalization is an extremely interesting but relatively poorly understood process in quantum statistical mechanics [102]. In holographic CFTs we obtain a lot of intuition as to how thermalization manifests as it has a dual process in terms of black hole collapse.

The potential reinterpretation of thermalization in terms of black hole collapse teaches us some things about 2d semi-classical CFT that are uncommon for quantum field theories. As was discussed to quite some length in chapter 3, the BTZ solutions of asymptotically AdS_3 gravity posses a mass gap. Hence collision processes, even at fairly high energies, can fail to form black holes. Furthermore the total energy of an initial scattering state is not necessarily indicative of whether a black hole will form as arbitrarily large amounts of energy can be locked up in terms of boundary charges. For 2d semi-classical CFTs this tells us that a high-energy atypical state can fail to thermalize despite the fact that other sectors of Hilbert space at the same or even lower energy will do so. As mentioned in [103][125] this separation of states in an ergodic phase and those in a thermalizing phase is quite uncommon in quantum field theory. Finally since gravitational collapse is a purely gravitational phenomena, the conventional AdS/CFT dictionary tells us that the identity block contribution to a correlator contains enough information to describe thermalization in the CFT.

The goal of this chapter will be to demonstrate a mechanism by which a class of out-of-equilibrium states settle down to equilibrium states. The assumptions that will come into play will be those of the previous chapter: large central charge, a sparse spectrum and the scaling conjecture. Under these assumptions it will be demonstrated that the problem of identifying which states will (not) thermalize is equivalent to the problem of determining a particular monodromy class of the solutions of the Fuchs equation from the previous chapter. Additionally two arguments will be given that advocate identifying the resulting late-time temperature with the Bekenstein-Hawking temperature of the post-collapse black hole.

5.0.1 Diagnozing thermalization from heavy states

The objects to study is the late-time behaviour of class of heavy non-equilibrium states $|V\rangle$. In this case that means that $|V\rangle$ has an order $\mathcal{O}(c)$ expectation value under the Hamiltonian L_0, but $|V\rangle$ will *not* be restricted to be an eigenstate of the Hamiltonian.

The natural observable to compute in order to probe the late-time dynamics of $|V\rangle$ is to calculate the expectation value of a class of 'simple' operators on this state. Simple operators are operators or products of operators that create low-energy excitations, they are insensitive to the microscopic details of the state and hence unable to resolve them. As such they naturally yield the thermodynamic properties of the state. This intuitive picture culminates in the Eigenstate Thermalization Hypothesis (ETH) [102]. The ETH is an operator statement concerning the matrix components of simple operators in energy eigenstate basis. In this basis the diagonal components correspond to the thermal expectation values of the operator in a thermal ensemble with an internal energy matching the energy eigenvalue associated to the matrix component. All other matrix components are proportional to $\exp(-S/2)$ where S is the entropy at the appropriate internal energy, furthermore these components are modulated by randomly distributed phases. These random phases ensure that the subdominant components will not conspire to cause deviations away thermal equilibrium with the exception of Poincaré recurrences.

In particular the simple operator that will be considered in this chapter is the product of two light primary operators with scaling dimensions of order $\mathcal{O}(c^0)$. Hence the Lorentzian correlator under consideration is

$$\langle V|Q(t_1)Q(t_2)|V\rangle. \tag{5.1}$$

It will be said that $|V\rangle$ will have thermalized if at late Lorentzian time the expectation value of the probe operators approaches its thermal expectation value. Or in order to quantify that statement

$$\langle V|Q(t_1)Q(t_2)|V\rangle \overset{t_1,t_2\gg 1}{=} \mathsf{Tr}(e^{-\beta H}Q(t_1)Q(t_2)) + \mathcal{O}\left(\frac{1}{c}\right), \tag{5.2}$$

with some appropriate value for the inverse temperature β. In the case of the two primary probes the thermal expectation value is in fact known [133]

$$\mathsf{Tr}(e^{-\beta H}Q(t_1)Q(t_2)) = e^{-2\pi h_Q(t_1-t_2)/\beta}. \tag{5.3}$$

After covering how to contruct Lorentzian out-of-time ordered correlators out of Euclidean correlators it will be shown that under certain conditions the Lorentzian correlators start to converge to display thermal behaviour at late time. The relevant condition is that the solutions to the Fuchsian equation of the last chapter satisfy certain monodromy properties.

The states

The heavy states under consideration will be quite broad but not all encompassing. In order to describe a semi-classical collapsing geometry we need our state to satisfy two properties

- The stress tensor expectation value $T(z) = \langle V|T(z)|V\rangle/\langle V|V\rangle$ needs to inherit the trasnformation rule of the stress tensor. This ensures that the state describes a single bulk Bañados geometry (i.e. no quantum superposition of geometries).

- There should exist an OPE channel where the identity block exchange dominates. This ensures that bulk interaction is dominated by the gravitational interaction, and secondly because it greatly simplifies the resulting analysis.

The first of these constraints can be met by considering states that are constructed by products of heavy operators acting on the vacuum

$$|V\rangle = O(x_1)...O(x_n)|0\rangle. \tag{5.4}$$

This is due to the fact that the stress tensor is a light operator and hence illustrates the decoupling property of light operators (4.4). The second constraint can be met by an assumption on the operator locations on the Euclidean radial plane. The assumption is that the insertion points x_i are located within a thin annulus whose exterior border touches the unit circle. The adjoint operators generating the bra-state $\langle V|$ are reflected around the unit circle (see (2.70)). Hence the OPE channel where the identity block dominates is the one where the 'mirror pairs' of operator and its adjoint are contracted by the OPE. A pictorial representation of the types of heavy operator distributions under consideration is given in figure 5.1

the correlators

Besides the heavy operators the correlation function (5.1) contains probe operators. As these probes are light their dynamical contribution $Q(z_1, z_2)$ to the conformal block factors out of the heavy sector. In chapter 4 a strategy for obtaining this factorized part of the identity block was presented which was related to the uniformization of punctured Riemann surfaces. The presented conclusion was that

$$Q(z_1, z_2) = \psi(z_1)^{-2h_Q}\psi(z_2)^{-2h_Q}\left(\int_{z_1}^{z_2} dz\,\psi(z)^{-2}\right)^{-2h_Q}, \tag{5.5}$$

where the function $\psi(z)$ corresponds to the general solution of the linear differential equation

$$\psi''(z) + \frac{6}{c}T(z)\psi(z) = 0. \tag{5.6}$$

Throughout the rest of this chapter the full correlator will be equated to this conformal block. The main claim of this chapter is that as long as one is concerned with the late time behavior of the Lorentzian correlator (5.1), one can

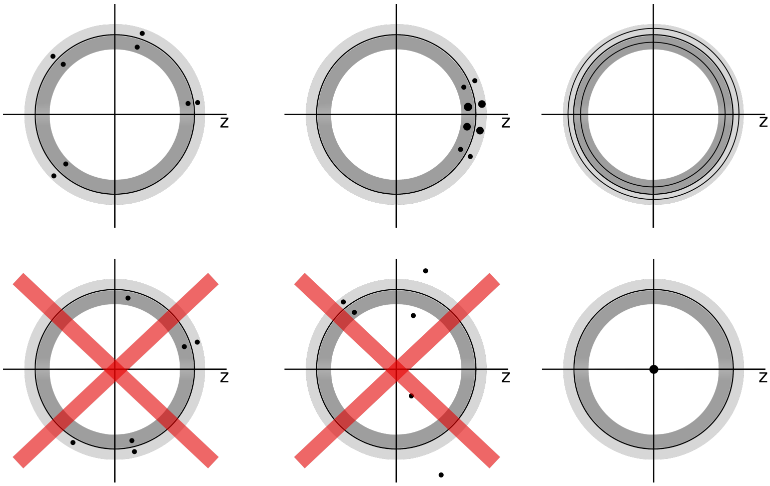


Figure 5.1: The types of heavy operator distributions under consideration. The ket state operators are restricted to the annulus cradling the unit circle. The adjoint operators creating the bra state are reflected around the unit circle. Two special cases are highlighted in the rightmost column, one is the continuous state considered in [104], the second is the (trivial) energy eigenstate. The crossed out diagrams provide examples of configurations that fall outside of our scope.

actually get around this problem of finding explicit solutions to the uniformization problem which in this case amounts to solving a second order differential equation of Fuchsian class. It will be shown that most of the relevant information about the late time thermalization is already encoded in the monodromy matrix of the solutions along a curve traversing the unit circle. In particular the temperature of the final state is related to the eigenvalue of the monodromy matrix. However, finding the monodromy matrix explicitly for a general Fuchsian equation is technically as daunting as solving the original uniformization problem. In this chapter we introduce two completely different methods to establish the precise connection between the eigenvalues of the monodromy matrix at late Lorentzian time and the final black hole temperature even without knowing the explicit matrix.

The first approach relies on finding a conformal transformation that preserves the structure of the monodromy matrix while leading to a new Fuchs equation which is much easier to solve. In fact this transformation leads to a stress tensor expectation value similar to one obtained in the continuum limit of [104] discussed earlier. For the hyperbolic class of monodromy matrices, this yields a precise relation between the eigenvalue of the monodromy matrix and the Hawking temperature of the black hole in the late Lorentzian time limit.

The second approach exploits the representation of the monodromy matrix in terms of a path-ordered integral over a flat connection [115]. We then apply the Chern-Simons formulation of pure gravity to relate this path ordered integral to the area of horizon of the black hole. Upon using the Bekenstein-Hawking formula this yields the same relation between the eigenvalue of the monodromy matrix and the Hawking temperature of the black hole after collapse.

The first of our methods tells us something more about the final state of the Lorentzian time evolution. In fact, at late Lorentzian time, one generically obtains a black hole dressed with soft gravitational hair. This soft hair corresponds to "boundary gravitons", namely some non-propagating graviton degrees of freedom localized near the asymptotic boundary of the AdS_3 spacetime. From the perspective of the dual CFT, these modes can be understood as acting with a raising Virasoro generator which affects the energy of the state, however, without modifying the temperature associated with the state. Therefore only looking at the temperature of the final state it is not possible to distinguish between a black hole state and a black hole state dressed with boundary graviton modes. The conformal transformation we used in our first approach has the holographic interpretation of a (holonomy-preserving) large diffeomorphism that precisely chops off this hair to yield a pure black hole as the final state of collapse.

Additionally it will be shown that possibility to lock arbitrarily large amounts of energy in the boundary charges is not just a mathematical curiosity. By numerical means it will be demonstrated that under a natural class of initial states the mismatch between the initial energy and late-time black hole mass can be of order $\mathcal{O}(c)$.

5.1 Lorentzian time-evolution on the radial plane

The eventual goal of this chapter is to describe the dynamics of probe correlators at asymptotically late Lorentzian times. This matter is complicated by the fact that the probe correlators

$$\langle V(0)|Q(t_1)Q(t_2)|V(0)\rangle = \langle V|e^{-iH(t_1-t_{\langle V|})}Qe^{-iH(t_2-t_1)}Qe^{-iH(t_{|V\rangle}-t_2)}|V\rangle \tag{5.7}$$

are out of time order[1]. A consequence of the Osterwalder-Schrader reconstruction theorem of quantum field theory [120, 121] states that all operator orderings within a correlator are related to each other through analytic continuation [122]. Hence the problem is simply to take the correlator involving the conventional operator ordering, analytically continue the domain of the insertion locations and evaluate the resulting function on the correct Riemann sheet. The Heisenberg picture above elucidates that the correlator will have a favorable decay property on the extended domain if the *time-differences* are continued to the lower half-plane. Hence on the extended domain the imaginary parts of the times have to satisfy the following list of inequalities

$$\mathrm{Im}(t_{\langle V|}) > \mathrm{Im}(t_1) > \mathrm{Im}(t_2) > \mathrm{Im}(t_{|V\rangle}). \tag{5.8}$$

In this case we will give the Lorentzian times small imaginary parts, which can equivalently be thought of as giving them small real shifts in Euclidean time instead. This suggests the following algorithm[2]

- Give all Lorentzian times imaginary parts such that the time differences fall within the appropriate extended domain.

[1] For notational convenience we gave the state $|V(0)\rangle$ a single time label. As mentioned in the previous section we do not want to restrict ourselves to heavy operators inserted at the same time, therefore the appropriate way to think of $t = 0$ is as the time of the latest heavy operator inserted and the effect of all other heavy operators on the state $|V(0)\rangle$ is included through a path integral.

[2] See [123] for an explicit demonstration of Lorentzian continuation of Euclidean correlators in a similar context.

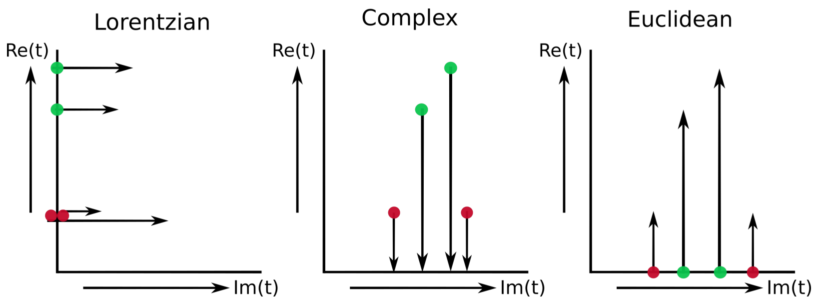


Figure 5.2: A pictorial representation of the analytic continuation to Lorentzian time. First give the time coordinates imaginary parts in compliance with (5.8). Subsequently set all real parts equal to zero resulting in an Euclidean correlator. Evaluate the Euclidean correlator and as a final step continuously shift all real parts back to their original values. This provides a prescription to find the right Riemann sheet.

- Set all real parts of the Lorentzian times equal to zero, the resulting correlator is purely Euclidean and can be computed using standard CFT techniques.
- As a final step bring all real parts of Lorentzian time (or equivalently imaginary parts of Euclidean time) back to their original values. The imaginary parts will prescribe how to circumnavigate all encountered branch points and the associated multi-valuedness of the Lorentzian correlator.

This method provides our strategy to deriving results at late Lorentzian time, a pictorial representation of the algorithm is presented in figure 5.2. As mentioned before, we will evaluate our correlator within radial quantization. We make use of the exponential map to see how our shifts in Euclidean time manifest on the radial plane.

$$z = e^{\tau + i\phi}, \qquad \bar{z} = e^{\tau - i\phi}.$$

One can see that a small shift in Euclidean time corresponds to a small radial shift on the radial plane. This explains the σ_i regularization scheme presented in the last section. If the heavy state operators and their adjoints are respectively shifted radially away from or toward the origin while maintaining the position of the probe operators on the unit circle we ensure that we are on the right branch after reinstating the appropriate Lorentzian times. Hence

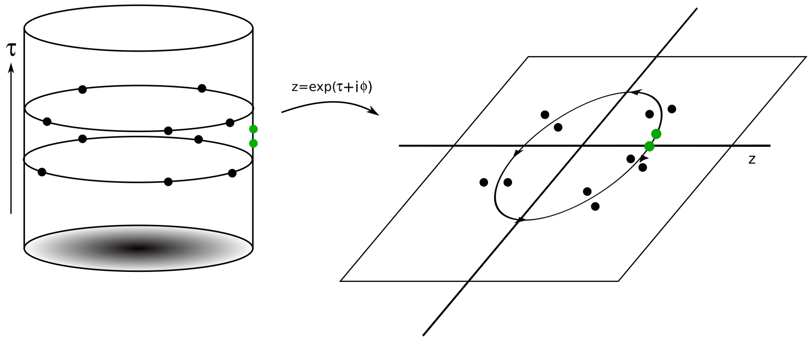


Figure 5.3: This diagram illustrates the effect of the exponential map on states created on the Euclidean cylinder. The black dots correspond to the insertions of heavy operators while the light dots denote light probe operators.

from then onwards Lorentzian time-evolution of the probe correlators simply corresponds to letting them perform cycles along the unit circle.

This implies that this cycle needs to possess a non-trivial monodromy in the case where the correlation function is not expected to be periodic in time. Let us return to the Fuchsian equation (2.117), it is a second-order linear differential equation which means that its general solution is a linear combination of two particular basis solutions

$$u(z) = c_1 u_1(z) + c_2 u_2(z). \tag{5.9}$$

Due to the presence of regular singular point in $T(z)$ on the complex plane the linear space spanned by the particular solutions possesses a non-trivial flat connection on the unit circle. This statement can be made visible by rewriting the single second-order ODE (2.117) into two coupled first-order ODEs[115]

$$\partial_z U(z) = a(z) U(z), \tag{5.10}$$

here $U(z)$ is the fundamental matrix of solutions and $a(z)$ is the connection matrix

$$U(z) = \begin{pmatrix} u_1(z) & u_2(z) \end{pmatrix}, \qquad a(z) = \begin{pmatrix} 0 & 1 \\ -\frac{6}{c}T(z) & 0 \end{pmatrix}. \tag{5.11}$$

In principle the general solution of this set of ODEs can be written as an initial value problem

$$U(z) = \mathcal{P}\left\{ e^{\int^z dz' \, a(z')} \right\} U(0), \tag{5.12}$$

here $\mathcal{P}$ designates the path-ordered integral. Hence the monodromy relevant for Lorentzian time-evolution is given by

$$M \sim \mathcal{P}\left\{e^{\oint_{|z|=1} dz\, a(z)}\right\}, \tag{5.13}$$

here $\sim$ indicates equal up to a similarity transformation. From the fact that $a(z)$ is always traceless we can establish the general property that the monodromy matrix M possesses unit determinant, i.e. as a matrix M in contained within the group $SL(2,\mathbb{C})$. In principle one can go a step further, the symmetric distribution of heavy operators and their adjoints around unit circle provides the following reflection property

$$T(z) = z^{-4}\overline{T(1/\bar{z})}. \tag{5.14}$$

As rather elegantly proven in [97], this property implies that in general the monodromy matrix M is up to a similarity transformation contained within the group $SU(1,1)$. Since similarity transformation do not effect the eigenvalues this restricts the eigenvalues of M to be either pure phase or purely real (see figure 5.4). This divides M into one of three classes, M either has: purely real eigenvalues, purely imaginary eigenvalues or M finds itself at one of the two cross-over point at 1 and -1 where the eigenvalues are degenerate.

The path-ordered integral over the flat connection above has a natural interpretation in the Chern-Simons formulation of gravity in AdS_3. We will exploit this interpretation later on in section 5.2.

5.1.1 Late-time behavior of the correlators

It is interesting to see what kind of algebraic effect the considerations of the previous section have on (5.1) in the late-time limit obtained by letting the probe correlator undergo a large number of cycles on the unit circle. Let us assume a basis of solutions that diagonalizes the monodromy matrix around the unit circle[3]. Since the monodromy has to have unit determinant, it only has one free eigenvalue λ, the other one has to be its inverse $1/\lambda$. After n cycles the general solution to the Fuchs equation can be decomposed as

$$u\left(e^{2\pi i n} z\right) = \lambda^n c_1 u_1(z) + \lambda^{-n} c_2 u_2(z), \tag{5.15}$$

we here assume, without loss of generality, that $|\lambda| \geq |1/\lambda|$. There are two particular cases of interest. Either both eigenvalues are distinct and pure

[3] This can be done whenever the eigenvalues are non-degenerate, which are our cases of interest

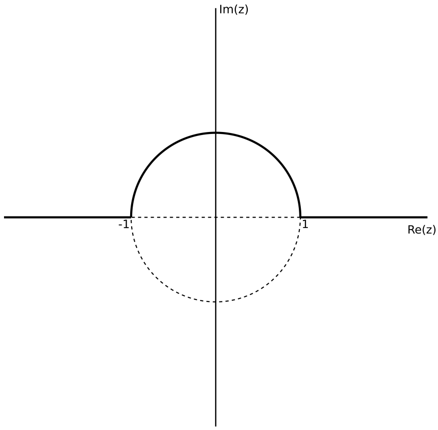


Figure 5.4: The restriction of the unit circle monodromy matrix to $SU(1,1)$ restricts the eigenvalues it can possess to be either purely real or pure phases. Since the eigenvalues are each others inverse $\lambda_1 = 1/\lambda_2$, the domain in which the eigenvalues can reside can be further split up into two independent domains indicated by the solid line and the dashed line.

phase, in which case time-evolution in Lorentzian time will yield (almost) periodic correlation functions. The physical interpretation is that the CFT state $|V\rangle$ is contained within an ergodic section of the phase diagram. The bulk dual statement is that the system fails to collapse into a black hole. The monodromy falls into elliptic class by the classification in [81].

The second case is when $|\lambda| > 1$, which is most interesting scenario. Since n plays the role of a discretized time, increasing n results in higher powers of the monodromy matrix. This is essentially a Markov chain process whereby the dominant eigensolution gets projected out as n increases. The dominant $u_1(z)$ mode acts as a stable mode, whereas $u_2(z)$ plays the role of a decaying mode. We find that the mode $u_1(z)$ should be given by a solution that corresponds to a BTZ black hole, since this is the situation that the collapse process should converge to at late times. Furthermore if both z_1 and z_2 make the same number of cycles the constant λ drops out of the correlator (5.1), this memorylessness property completes the argument that this process is essentially Markovian. As we will see now, it also corroborates the interpretation we will pursue, which is that this is essentially a thermalization process.

The eigenvalue is not purely a mathematical construct, it can be given a physical interpretation by considering the late-time limit. Consider $z_1 = e^{2\pi i n} z_0, ;\ z_2 = e^{2\pi i (n+m)} z_0$, where z_0 is some number on the unit circle on the principle sheet and $n \gg 1$, the integer separation of the times is purely for notational convenience, we will take m to be large soon after which the remainder under a more general treatment would be a subdominant correction. By allowing z_2 to make a few more cycles compared to z_1, corresponding to time-advancing $Q(z_2)$ we can obtain the following result for the correlator

$$\begin{aligned}
&\langle O(x_1)...O(x_n)Q(z_1)Q(z_2(n))\rangle = u(z_1)^{-2h_Q}\lambda^{-2mh_Q}u(z_2)^{-2h_Q} \\
&\times\left(\int_{z_1}^{e^{2\pi i}z_1} u(z)^{-2}\,dz + \int_{e^{2\pi i}z_1}^{e^{4\pi i}z_1}\lambda^{-2}u(z)^{-2}\,dz + ... + \int_{e^{2\pi i(m-1)}z_1}^{e^{2\pi i m}z_1}\lambda^{-2(m-1)}u(z)^{-2}\,dz\right)^{-2h_Q} \\
&= u(z_1)^{-2h_Q}u(z_2)^{-2h_Q}\lambda^{-2mh_Q}\left(\frac{\lambda^{-2m}-1}{\lambda^{-2}-1}\right)^{-2h_Q}\left(\int_{z_1}^{e^{2\pi i}z_1} u(z)^{-2}\,dz\right)^{-2h_Q}.
\end{aligned} \tag{5.16}$$

Hence as m becomes larger we note the approximate scaling law

$$\langle O(x_1)...O(x_n)Q(z_1)Q(z_2(m))\rangle \sim \lambda^{-2mh_Q}. \tag{5.17}$$

It is worth noting that the scaling behaviour is independent of n and only depends on the difference between the number of cycles which can be interpreted as the difference in time when the two operators are fixed at the same spatial point. Such a scaling relation at late times is suggestive of thermal behavior. In the upcoming sections we will support this observation by two separate arguments. Firstly we will demonstrate that when the eigenvalues are purely real we can construct a monodromy preserving diffeomorphism to a stress tensor whose associated Fuchs equation can be solved.

Secondly we will demonstrate that the eigenvalue λ has a natural manifestation in the Chern-Simons formulation of $3d$ Einstein gravity, as being related to the temperature of the bulk black hole geometry associated to the boundary connection $a = \left(J_{-1} + \frac{6}{c}T(z)J_1\right)dz$, where the matrices J_i are generators of $SL(2,\mathbb{R})$. Either way we will find out that the explicit relationship is given by

$$\log(|\lambda|) = 2\pi^2 T_{BH}, \tag{5.18}$$

with the black hole temperature T_{BH}. If we equate $2\pi m$ with time difference $(t_2 - t_1)$ we find the following asymptotic scaling

$$A_0(t_1,t_2) \sim e^{-2\pi h_Q T_{BH}(t_2-t_1)}, \tag{5.19}$$

where we have put the two operators at the same spatial point. This is exactly the form of a thermal correlator with temperature T_{BH}. This provides our general conclusion. Without strictly solving the Fuchs equation we find that at late Lorentzian time, under certain conditions, the probe correlator converges to a thermal correlator with a temperature identified with the temperature of the bulk theory black hole.

5.1.2 Monodromy preserving diffeomorphisms and their bulk interpretation as boundary gravitons

In this section we provide the first argument that the hyperbolic class of asymptotic Lorentzian correlators correspond to thermal black hole correlators. The proof provided here is the most straightforward of the two, but also the most abstract. As a consequence it clarifies that the bulk final state consists of a black hole dressed by boundary gravitons. Here we will exploit the fact that the stress tensor $T(z)$, for which the associated monodromy M has real eigenvalues, must be contained within a specific class of Virasoro coadjoint orbits [54, 124]. Different points in the orbit are related by conformal transformations. In the dual gravity picture, this orbit can be associated with a class of black hole solutions to the Einstein equations.

As was briefly reviewed in chapter 3, any asymptotically AdS$_3$ solution can be written as a Bañados geometry [66]

$$ds^2 = \frac{dr^2}{r^2} - \left(r dz - \frac{\bar{L}(\bar{z})}{r} d\bar{z}\right)\left(r d\bar{z} - \frac{L(z)}{r} dz\right), \tag{5.20}$$

where $L(z)$ and $\bar{L}(\bar{z})$ are single-valued functions related to the holomorphic and anti-holomorphic stress tensors of the boundary CFT. After fixing the Bañados gauge there exists a residual symmetry that preserves the general form of this metric but changes the functions $L(z)$ and $\bar{L}(\bar{z})$. At $r \to \infty$ the vector fields generating these coordinate transformations are given by [18]

$$\zeta = \left(\xi(z) + \frac{1}{2r^2}\partial_{\bar{z}}^2\bar{\xi}(\bar{z})\right)\partial_z + \left(\bar{\xi}(\bar{z}) + \frac{1}{2r^2}\partial_z^2\xi(z)\right)\partial_{\bar{z}} - \frac{r}{2}\left(\partial_z\xi(z) + \partial_{\bar{z}}\bar{\xi}(\bar{z})\right)\partial_r. \tag{5.21}$$

The set of all these vector fields is parametrized by the two functions $\xi(z)$ and $\bar{\xi}(\bar{z})$. Under the infinitesimal coordinate transformations generated by these vector fields the functions $L(z)$, $\bar{L}(\bar{z})$ transform according to the infinitesimal Virasoro Ward identity

$$\delta_\zeta L(z) = 2L(z)\partial_z\xi(z) + \xi(z)\partial_z L(z) - \frac{1}{2}\partial_z^3\xi(z), \tag{5.22}$$

with a similar rule holding for $\bar{L}(\bar{z})$. By identifying the conformal charges with the Brown-Henneaux charges the function $T(z)$ was identified with the functions parametrizing the Bañados metric through $L(z) = \frac{6}{c}T(z)$, see chapter 3 for an extensive review. From this one can conclude that the dual description of (single-valued) conformal transformations is given by large diffeomorphisms that preserve the asymptotic boundary conditions of the AdS_3. Hence stress tensors that are connected to each other through single-valued coordinate transformations correspond in the bulk to geometries that are related to each other through boundary gravitons.

Given a function $T(z)$ (correspondingly $L(z)$) it is natural to consider the set of all other functions $T(z)$ that can be obtained by single-valued conformal transformation. To remind ourselves the infinitesimal transformation law integrates to one of the form of (2.49), namely,

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z(w)) + \frac{c}{12}S[z(w), w], \tag{5.23}$$

for a finite conformal transformation. The orbits of $T(z)$ under all single-valued conformal transformations is directly related to the Virasoro coadjoint orbits discussed in the second chapter of this thesis. The Virasoro orbits have all been classified in terms of the monodromy of the solutions Hills equation [124] (see table 2.2). Up to a change of variables Hill's equation is the Fuchs equation whose domain has been restricted to the unit circle, this will be elaborated upon in the next chapter. Each orbit is classified by a reference point, a function $T(z)$, usually of some convenient form, that can be reached through means of conformal transformation.

Of special interest are the the functions $T(z)$ that generate solutions to the Fuchs equation whose monodromy matrix has real eigenvalues, these coadjoint orbits contain a reference point which are the CFT duals of the standard BTZ geometries [70, 65, 71]. From the CFT perspective this reference point corresponds to a stress tensor expectation value associated to a state created by acting on the vacuum with a primary operator whose scaling dimension satisfies $H > c/12$.

From the boundary CFT perspective it is not entirely obvious that the generic stress tensor we are considering is contained within an orbit that contains as a reference point a stress tensor associated to a state created by a single primary operator. First of all, it would naively be expected that a generic heavy CFT state would correspond to a linear combination of energy eigenstates. However, on the other hand, the existence of the BTZ reference point suggests that even a generic heavy stress energy function is dominated

Class	eigenvalues	Coadjoint orbit	dual geometry
Elliptic	pure phase $(\lambda \neq 1, -1)$	$\mathcal{C}_\nu \quad 0 < \nu < 1$	Conical defect
Parabolic	$\lambda = 1$ or -1	$\mathcal{P}_0^+$, $\mathcal{P}_1^-$	minimal mass BTZ
Hyperbolic	purely real	$\mathcal{B}_0(b)$	BTZ black hole
Exceptional	$\lambda = 1,$	$\mathcal{E}_1$	Vacuum

Table 5.1: A characterisation of the relevant monodromy classes their associated Virasoro coadjoint orbit and the geometry of the holographic dual. (following [124],[81])

by contributions coming from a single conformal family. These two apparently conflicting CFT statements can actually be justified in the large c limit. In appendix C we have provided an explicit example towards the justification. There we consider a state generated by n heavy operators inserted on some radial slice. As the energy of our state increases the variance in the energy increases as well. However, the variance increases parametrically less fast than the expectation value ($\sqrt{nh/\sigma}$ as opposed to nh/σ). As one can check, for a heavy state with an energy expectation at least of the order of the central charge the energy distribution can be well considered as being sharply peaked. This is of course directly analogous to the intuition of the eigenstate thermalisation hypothesis [102]. Therefore the general expectation is that a pure state created by acting with the heavy primary operators on the vacuum is dominated by the contribution of a single energy eigenstate.

Secondly, of course, a generic $T(z)$ does not correspond to a primary state, in fact in general it will be a generic element of the Virasoro orbit associated to a primary state. In principle all states within a Virasoro orbit are connected to each other by conformal transformations. So it remains to determine which primary state is associated to the conformal family that our heavy state $|V\rangle$ actually belongs to. This primary state should be related to our heavy state by some judicially chosen single-valued conformal transformation [125]. The Fuchs equation mirrors this picture; the conjugacy class of the monodromy matrix around the unit circle remains invariant under the group of orientation preserving diffeomorphisms of the circle $\mathrm{Diff}_0(S^1)$ [124]. This is due to the eigenvalue being related to the scaling weight of the primary state and is hence an orbit invariant. We will now construct an element of $\mathrm{Diff}_0(S^1)$ similar to the one in [124] that brings us to the stress tensor of a single primary operator inserted at the origin with a specific scaling dimension. We can connect to the intuition above by interpreting the diffeomorphism as the transformation that undoes the boundary gravitons. The Fuchs equation is given by

$$\psi''(z) + \frac{6}{c}T(z)\psi(z) = 0. \tag{5.24}$$

We assume that $T(z)$ is such that the monodromy along the unit circle of solutions falls in the hyperbolic class corresponding to real eigenvalues with $|\lambda| > 1$. A complete classification of the monodromy matrices is presented in Table 5.1. Consider the eigenbasis of solutions under the monodromy transformation along the unit circle

$$\psi_1\left(e^{2\pi i}z\right) = \lambda\psi_1(z), \qquad \psi_2\left(e^{2\pi i}z\right) = \frac{1}{\lambda}\psi_2(z). \tag{5.25}$$

In this case it is clear that the ratio of these two solutions $f(z) = \psi_1(z)/\psi_2(z)$ has the following transformation property

$$f\left(e^{2\pi i}z\right) = \lambda^2 f(z). \tag{5.26}$$

From this transformation rule we can construct a function that is inherently single-valued on the unit circle

$$u(z) = e^{\frac{i\pi}{\gamma}\log(f(z))} = f(z)^{\frac{i\pi}{\gamma}}, \tag{5.27}$$

since this function simply adds a term $2\pi i$ to the exponent after making a full circle. Here γ is defined by $\gamma = \log(|\lambda|)$. To prove that the function above is an element of $\mathrm{Diff}_0(S^1)$ we need to show that it is smooth everywhere on the unit circle. The assumption that our stress tensor is an element of a hyperbolic orbit and that the zero-mode of the stress tensor is bounded from below restricts the orbit to the coadjoint orbit $B_0(b)$, as per the classification given in [124]. The complete classification is briefly summarized in table 2.2. It is known that if $T(z)$ is contained within $B_0(b)$, the eigenbasis of solutions to the Fuchs equations has no roots along the unit circle [65, 71]. This implies that the ratio of solutions $f(z)$ is forced to be smooth on the unit circle. Therefore we can conclude that $u(z) \in \mathrm{Diff}_0(S^1)$. Next, with some algebra we will show that this function, $u(z)$ has a desirable property.

Taking the Schwarzian derivative of $u(z)$ and making use of the group structure of the Schwarzian derivative results in

$$S[u(f(z)), z] = S[f, z] + \left(\frac{df}{dz}\right)^2 S[u, f]. \tag{5.28}$$

From the theory of Fuchs equations we know that under the Wronskian normalization condition $\psi_2(z)\psi_1'(z) - \psi_1(z)\psi_2'(z) = 1$ the following identity holds

$$S[f,z] = \frac{12}{c}T(z), \tag{5.29}$$

This is of course the Schwarzian equation discussed in chapter 4. Inserting this identity in (5.28) and rearranging the terms a little bit yields

$$\begin{aligned} T(z) &= \left(\frac{du}{dz}\right)^2 \frac{c}{24\pi^2}\left(\gamma^2+\pi^2\right) u(z)^{-2} + \frac{c}{12}S[u,z] \\ &\equiv \left(\frac{du}{dz}\right)^2 \tilde{H}\, u(z)^{-2} + \frac{c}{12}S[u,z] \end{aligned} \tag{5.30}$$

This is, therefore, exactly the coordinate transformation that yields a uniform stress component of the form $\tilde{H}/u^2$, where the role of the scaling dimension is now played by

$$\tilde{H} \leftrightarrow \frac{c}{24\pi^2}\left(\gamma^2+\pi^2\right), \tag{5.31}$$

Note that the scaling dimension has a minimum value given by $c/24$, this is enforced by the initial assumption that we are in a hyperbolic monodromy class, the minimum value correponds to the zero mass BTZ state in the bulk. Given the primary stress tensor the monodromy matrix can be computed by explicitly calculating the path-ordered integral [85]. But the associated Fuchs equation can also be solved explicitly. In fact in [94] the authors found uniformized correlator associated to these solutions and find that they produce a thermal correlator with temperature $T_H = \frac{1}{2\pi}\sqrt{24H/c-1}$, inserting (5.31) and reinstating λ through $\gamma = \log(|\lambda|)$ gives us the relationship

$$|\lambda| = e^{2\pi^2 T_H}, \tag{5.32}$$

quoted in the previous section.

5.2 Chern-Simons interpretation of the monodromy

As shown in the previous section, the late-time behavior of the probe correlation function is controlled by a monodromy matrix. In this section we will demonstrate that this matrix has a natural manifestation in the Chern-Simons formulation of $3d$ gravity that was briefly reviewed at the end of the third chapter. The discussion in this section is heavily based on [85], additional references include [125, 126, 128, 129]. The topological nature of $3d$ gravity is most clearly expressed in terms of so(2,2) connection field A [130],

$$\mathbf{A}_i = e_i^a \mathbf{P}_a + \omega_i^a \mathbf{J}_a, \tag{5.33}$$

here e_i^a and ω_i^a are respectively the vielbein and the spin connection associated to the Einstein-Hilbert action. P_a and J_a denote generators for translation and Lorentz transformations respectively. After imposing that A transforms as a non-abelian gauge field under local $SO(2,2)$ transformations it can be shown that the Chern-Simons action

$$S_{CS}\left[\mathbf{A}\right] = \frac{k}{4\pi}\int \mathrm{Tr}\left(\mathbf{A}\wedge d\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right), \tag{5.34}$$

is equivalent to the Einstein-Hilbert action with negative cosmological constant. The constant k is the level of the Chern-Simons theory. By means of the Brown-Henneaux formula it can be related to the central charge through $k = \frac{c}{6}$. It turns out to be convenient sometimes to use the decomposition of the gauge group $SO(2,2)$ as $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. The connection field $\mathbf{A}$, accordingly, also assumes a decomposition in terms of a pair of $SL(2,\mathbb{R})$ connections, $\{A, \bar{A}\}$ defined through

$$A_i = \left(\omega_i^a + \frac{1}{R}e_i^a\right) J_a^{(+)}$$

$$\bar{A}_i = \left(\omega_i^a - \frac{1}{R}e_i^a\right) J_a^{(-)}$$

where $J_a^{(\pm)} = \frac{1}{2}\left(J_a \pm R P_a\right)$. With this decomposition the Einstein Hilbert action takes the form

$$S_{\mathsf{EH}} = S_{CS}\left[A\right] - S_{CS}\left[\bar{A}\right], \tag{5.35}$$

and the constant, R receives a natural interpretation as the radius of the AdS$_3$.

5.2.1 Wilson loops in Chern-Simons theory

In [85] Wilson lines were considered as geometrical probes for the bulk gravitational theory which act as the Chern-Simons analogue of massive probe particles traveling along geodesics in the bulk geometry. To make this statement quantitative

$$W_{\mathcal{R}}(C) \sim e^{-mL}, \tag{5.36}$$

the $\sim$ designates that this is an on-shell relation. Here C designates the curve along which the Wilson line is defined on the boundary of AdS$_3$. m is the mass of the relevant probe and L as the proper length of the geodesic connecting the endpoints of the Wilson line supported at the boundary. The subscript $\mathcal{R}$ denotes the $SL(2,\mathbb{R})$ representation in which the Wilson line falls. In order to allow for a continuous mass parameter for the probe the representation needs to be infinite dimensional, e.g. a highest weight representation of $SL(2,\mathbb{R})$ [85]. As we will find, the relation to the geodesic length, L stated above will turn out to be extremely useful. Given an appropriate bulk geodesic, this length will be interpreted as the horizon of the diffeomorphism-equivalent black hole which in turn is to its temperature by means of the Bekenstein-Hawking formula.

The key point of [85] is that the expectation value of the Wilson line is given by the path integral over the following action

$$S(U,P,A,\bar{A})_C = \int_C ds\,\mathrm{Tr}(PU^{-1}D_sU) + \lambda(s)\left(\mathrm{Tr}(P^2) - \frac{1}{2}m^2\right). \tag{5.37}$$

The Hilbert space of the quantum system described by the auxiliary field U and its conjugate momentum P corresponds to the vector space given by the carrier space of an infinite dimensional representation of $SL(2,\mathbb{R})$. The parameter s parametrizes the curve C, and D_s denotes the covariant derivative with respect to s, defined as

$$D_sU = \frac{dU}{ds} + A_sU - U\bar{A}_s, \qquad A_s \equiv A_\mu \frac{dx^\mu}{ds}. \tag{5.38}$$

We can choose the field U to live in the fundamental representation, this way we will connect to our monodromy matrix. Note that we are only integrating over the auxiliary fields and consider the Chern-Simons field A as a background field which fixes the background geometry, i.e. there is no gravitational backreaction. As a consequence, the probe mass can take any real positive value without affecting the dynamics. The equations of motion of this action are

$$U^{-1}D_sU + 2\lambda(s)P = 0, \qquad \partial_sP + \left[\bar{A}_s, P\right] = 0, \tag{5.39}$$

while the Lagrange multiplier $\lambda(s)$ imposes the constraint

$$\mathrm{Tr}(P^2) = \frac{1}{2}m^2. \tag{5.40}$$

On shell, the action reduces to a very simple form

$$S_{\text{on-shell}} = -m^2 \int_C ds\, \lambda(s). \tag{5.41}$$

When solving the equations of motion it is natural to define the function $\alpha(s)$ through $d\alpha/ds = \lambda(s)$. In this case the action simply becomes

$$S_{\text{on-shell}} = -m^2 \Delta\alpha, \tag{5.42}$$

the point is now clear, we need to solve for $\Delta\alpha$ and equate it to L/m. Note that $\Delta\alpha$ only depends on the endpoints of the curve. We will show below that knowledge of the boundary conditions is sufficient to solve for $\Delta\alpha$.

5.2.2 The nothingness trick

We will here follow steps very similar to those presented in [85]. The equations of motion are simple to solve in the 'empty gauge', i.e. $A = \bar{A} = 0$, so we will solve the equations of motion in that particular gauge and then gauge transform the resulting solution to generate more solutions. One could ask, how many solutions are gauge connected to the empty gauge? The answer is that the bulk equation of motion of Chern-Simons theory imposes that A is locally flat, hence all solutions to the Einstein field equations have an associated connection field A that is gauge connected to the empty gauge. These gauge transformations will, however, be multivalued in general and hence not globally well-defined.

In the empty gauge the equations of motion are given by

$$U_0(s)^{-1}\partial_s U_0(s) + 2\lambda(s)P_0(s) = 0, \qquad \partial_s P_0 = 0. \tag{5.43}$$

These equations are easily solved, their general solutions are given by

$$U_0(s) = u_0 e^{-2\alpha(s)P_0}, \qquad P_0(s) = P_0, \tag{5.44}$$

here as before $d\alpha/ds = \lambda(s)$. Under $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ gauge transformations the auxiliary field variables transform as

$$U(s) \to L(s)U(s)R(s), \qquad P(s) \to R^{-1}(s)P(s)R(s), \tag{5.45}$$

where $L(s)$ and $R(s)$ are elements of the fundamental representation of $SL(2,\mathbb{R})$. In order to obtain the gauge transformation that brings us to the required connection, we need to find a transformation L and R such that

$$LdL^{-1} = A = b^{-1}ab + b^{-1}db = b^{-1}\begin{pmatrix} 0 & 1 \\ -\frac{6}{c}T(z) & 0 \end{pmatrix} b + b^{-1}db, \tag{5.46}$$

and similarly

$$R^{-1}dR = \bar{A} = b^{-1}\begin{pmatrix} 0 & 1 \\ -\frac{6}{c}\bar{T}(\bar{z}) & 0 \end{pmatrix} b + b^{-1}db. \tag{5.47}$$

Here a is the boundary connection defined as $a = \left(J_{-1} + \frac{6}{c}T(z)J_1\right)dz$. It is related to the full connection through $A = b^{-1}(a+d)b$, with $b = e^{\rho J_0}$. J_i denote $SL(2,\mathbb{R})$ generators with $i = \{1, 0, -1\}$. With some simple algebra it can be shown that a solution to the differential equation for L is given by

$$L = e^{-\rho J_0}\mathcal{P}\left\{e^{-\int_0^{s_f} a(z)ds}\right\}. \tag{5.48}$$

One can similarly demonstrate that

$$R = \mathcal{P}\left\{e^{\int_0^{s_f} \bar{a}(\bar{z})ds}\right\} e^{-\rho J_0}. \tag{5.49}$$

We know what gauge transformation we want, the trick will be to apply it to the boundary conditions of the field variables. We are interested in Wilson loops (i.e. we want to find a geodesic that closes in on itself), therefore we have to impose periodic boundary conditions to the field variables

$$U(0) = U(s_f), \qquad P(0) = P(s_f). \tag{5.50}$$

Using the gauge transformation (5.45) the boundary condition for P gives us

$$\left[P_0, R(s_f)R^{-1}(0)\right] = 0. \tag{5.51}$$

This condition implies that P_0 commutes with the path-ordered integral over the right-moving boundary gauge component. Similarly working out the boundary condition on U gives us

$$u_0^{-1}L(s_f)^{-1}L(0)u_0R(0)R(s_f)^{-1}, \tag{5.52}$$

where as an intermediate step we used the fact above that P_0 commutes with the product $R(s_f)R^{-1}(0)$. Plugging (5.48) and (5.49) in (5.52) yields

$$e^{-2\Delta\alpha P_0} = u_0^{-1}Mu_0\bar{M}^{-1}, \tag{5.53}$$

Here M is the same matrix as the monodromy matrix M defined in (5.13) in section 3. $\bar{M}$ is defined in an analogous way as the path-ordered integral over $\bar{A}$. Due to our restriction to scalar heavy operators (i.e. $\bar{h} = h$) the barred path-ordered integral gives

$$\bar{M} = \mathcal{P}\left\{\exp(\oint \bar{a}\left(\bar{T}(\bar{z})\right) ds)\right\} = \mathcal{P}\left\{\exp(\oint a\left(T(\bar{z})\right) ds\right\} = M^{-1}, \tag{5.54}$$

the reason for this being that from the perspective of the anti-holomorphic variable $\bar{z}$ the contour is followed with the opposite orientation, hence

$$e^{-2\Delta\alpha P_0} = u_0^{-1} M u_0 M. \tag{5.55}$$

We can use (5.51) to simultaneously diagonalize the left- and right-hand side. Denoting the matrix that diagonalizes them by V,

$$e^{-2\Delta\alpha P_0} = \left(u_0 V\right)^{-1} M \left(u_0 V\right) \bar{M}^{-1}. \tag{5.56}$$

As pointed out in [85], consistency between the left- and the right-hand side demands that $\left(u_0 V\right)^{-1} M \left(u_0 V\right)$ is a diagonal matrix. Therefore we can make the above expression cleaner by defining λ_M to be the diagonal matrix whose components are the eigenvalues of M

$$e^{-2\Delta\alpha P_0} = \lambda_M^2. \tag{5.57}$$

We can take the matrix logarithm on both sides, since all matrices involved are diagonal this reduces to the logarithms of the components

$$-\Delta\alpha P_0 = \log(\lambda_M) = \begin{pmatrix} \log(|\lambda|) & 0 \\ 0 & -\log(|\lambda|) \end{pmatrix}. \tag{5.58}$$

We know $\mathrm{Tr}(P_0) = 0$ because it has to live in the Lie algebra of $SL(2,\mathbb{R})$. Then from the constraint $\mathrm{Tr}(P_0^2) = \frac{1}{2}m^2$, the eigenvalues of P_0 are given by $\pm m/2$. Contracting both sides with the matrix

$$J_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{5.59}$$

gives us

$$-\frac{m}{2}\Delta\alpha = \log(|\lambda|). \tag{5.60}$$

Inserting the expression $\Delta\alpha = -L/m$ yields

$$\log(|\lambda|) = \frac{1}{2}L. \tag{5.61}$$

As mentioned before, we assume that this L is the proper length of the geodesic connecting the endpoints of the Wilson loop. Due to the topological nature of 3d gravity we can perform gauge transformations that correspond to continuously deforming the contour, the argument is that if there is a black hole in AdS_3 we can only continuously deform a contour to a minimum size measured by the holonomy. Hence L measures the proper size of the horizon of black hole in the center of AdS_3. Making use of the Bekenstein-Hawking formule $S = A/4G$ and the Brown-Henneaux formula $c = \frac{3}{2G}$ finally gives us

$$\log(|\lambda|) = 2\pi^2 T_H. \tag{5.62}$$

This is the relationship between the eigenvalues of the monodromy matrix M and the temperature of the final state bulk black hole claimed in (5.18).

5.3 An analytic example and numerical checks

In the preceding section it was established that the eigenvalues of a certain monodromy matrix govern the late Lorentzian time behaviour of probe correlation functions. In this section we will first consider an example, which is the smooth limit originally considered in [104]. The reason being that for this particular stress tensor the monodromy matrix can be computed explicitly. In particular we will be able to see a transition from conical defect to black hole.

Secondly we perform a numerical analysis of the monodromy problem for various natural polyhedral operator distributions and demonstrate that there can exist an order $\mathcal{O}(c)$ deficiency between the initial energy injected in the bulk and the mass of the final black hole state. We interpret this deficiency as a fraction of the initial energy being frozen out in boundary gravitons, conserved charges associated with the asymptotic Brown-Henneaux Virasoro algebra of AdS$_3$. This demonstrates that the ability of the boundary charges to absorb large amounts of energy is more than just an artifact that only occurs in exotic states.

5.3.1 The continuous limit as an example

One particular state in which the monodromy matrix can be computed explicitly is the isotropically collapsing shell of null dust constructed in [104]. This state is produced by acting on the vacuum with a ring of regularly distributed dust operators and taking the number of insertions to infinity. We will not compute the full probe correlator, instead we will use this specific state as an example to demonstrate the general principle described in the previous sections. We will show that the monodromy matrix eigenvalues are related to the temperature of the thermal state that the state would equilibriate to[4].

This isotropic shell state leads to a relatively simple expression for the stress tensor. It closely resembles the stress tensor expectation value of the primary state. The state on which the correlator is evaluated is given by

$$|V\rangle = \lim_{n\to\infty} \frac{1}{\mathcal{N}} \prod_{k=1}^{n} O(e_k)|0\rangle, \qquad e_k = (1-\sigma)e^{2\pi i(k-1)/n}. \tag{5.63}$$

In the aforementioned paper it was shown that the stress tensor expectation value for this particular state can be written as

$$T(z) = \frac{K}{z^2}\Theta(|z|-1+\sigma)\theta(1-|z|+\sigma), \tag{5.64}$$

where K is a constant related to the CFT data through $K = \frac{H}{\sigma}$ and $\theta(z)$ denotes the heaviside function. It can easily be checked that this state has no gravitational hair, the reason being that along the unit circle stress tensor takes the form of the lowest energy element of either a $\mathcal{B}_0(b)$ or $\mathcal{C}_\nu$ Virasoro coadjoint orbit [124]. The stress tensor above is very similar to the one of an eternal black hole, the difference is the compact support of the above stress tensor. For any cycle contained within the annulus the only visible pole would be the double pole located at $z = 0$, for this reason it is possible to explicitly solve the path-ordered integral in (5.13) in a straightforward manner. Inserting this stress tensor expectation value into the flat connection (5.11) gives

$$a(z) = \begin{pmatrix} 0 & 1 \\ -\frac{6K}{c\,z^2} & 0 \end{pmatrix}. \tag{5.65}$$

The relevant information contained in the monodromy matrix are its eigenvalues. Hence we are only interested in its conjugacy class. We only need the connection up to an $\mathrm{SL}(2,\mathbb{C})$ gauge transformation. There exists a theorem that states that there always exists a gauge transformation such that around a singular point z_0 of $A(z)$ the connection can be transformed to a new connection $a(z) \to (z-z_0)^{-1-R_0}a_0(z)$, where the matrix elements of $a_0(z)$ are all regular functions of z [115, 127]. The integer power R_0 is the Poincaré rank of the singular point, $R_0 = 0$ corresponds to a regular singular point. The connection transforms under a gauge transformation in the usual way

$$a(z) \to UaU^{-1} + \left(\frac{d}{dz}U\right)U^{-1}, \tag{5.66}$$

[4]The final state can be a conical defect as well [125]

a simple proposal for a gauge transformation that brings the connection to minimal form is the matrix that locally diagonalizes the connection A(z), specifically

$$U(z) = \begin{pmatrix} -\frac{\sqrt{-\frac{3K}{2c}}}{z} & \frac{1}{2} \\ \frac{\sqrt{-\frac{3K}{2c}}}{z} & \frac{1}{2} \end{pmatrix}, \tag{5.67}$$

this matrix is the inverse of the matrix of eigenvectors of $a(z)$. Under this transformation it can be found that the irregular part in z (and in fact all z-dependence) factors out of the connection. The resulting path-ordered integral can be resolved by means of the following identity[115]

$$M = P\left\{e^{\oint dz\, \frac{1}{z} a_0(z)}\right\} = e^{2\pi i a_0(0)}. \tag{5.68}$$

The heuristic argument is that one can continuously deform the contour to a limiting circle around a pole, at which point the diagonalizing matrix becomes exactly constant, subsequently rendering the path-ordered integral into a normal matrix integral. Unfortunately this particular approach does not generalize to multiple poles in a straightforward manner, since one would need to consider integrating along paths connecting the poles. The eigenvalues of $2\pi i a_0(0)$ are given by

$$\{\lambda\} = i\pi \pm \pi\sqrt{\frac{24K}{c} - 1}, \tag{5.69}$$

from which we can conclude that

$$\{\lambda\}_M = -e^{\pm\pi\sqrt{\frac{24K}{c}-1}}. \tag{5.70}$$

In [104] it was established that $T_{BH} = \frac{1}{2\pi}\sqrt{24K/c - 1}$ therefore these eigenvalues are of the form derived in the previous sections. Which is exactly what was predicted above. One can observe both the the BTZ-mass threshold as well as the thermal behavior of the correlator at late time. This result is consistent with the statement made before that exact knowledge of the solutions of the Fuchs equation is over excessive for the purpose of finding the thermodynamic properties of the black hole at late time. Note that the analysis would have been identical for a state created by acting with a single primary operator at the origin on the vacuum.

5.3.2 Numerical results including boundary gravitons

Solving the path-ordered integral in a general setup is generally intractable. But since the path-orderd integral is the formal solution of an initial-value problem it is possible to draw some conclusions from numerical integration of this initial-value problem.

One potentially confusing aspect we will focus on is that there can be very large deviations between the intial energy injected into the system and the mass of the final black hole state. We denote by T'_H the temperature of a black hole in the hypothetical situation that all initial energy is converted into black hole rest mass. By applying formula (3.50) $T'_{BH} = \frac{\sqrt{3}}{\pi}\sqrt{E_L/c}$, one can construct the temperature of such a hypothetical black hole. Here the Lorentzian energy on the cylinder is given by

$$\frac{1}{2}E_L = \frac{1}{2\pi i}\left(-i\frac{\pi c}{12} + \oint dz\, zT(z)\right), \tag{5.71}$$

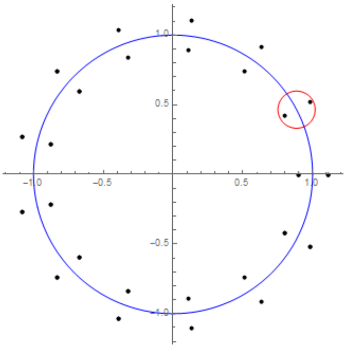
where the first term is the Casimir energy of the cylinder, which is also the minimal mass of the black hole [10]. The factor 1/2 on the left-hand side is to emphasize that one has to add the identical holomorphic and anti-holomorphic contributions. Combining the above with the expression for the eigenvalues 5.18 one finds that these are given by

$$\text{“}|\lambda'_\pm|\text{”} = e^{\pm 2\sqrt{3}\pi\sqrt{\frac{1}{i\pi c}(\oint dz\, zT(z)) - \frac{1}{12}}}, \tag{5.72}$$

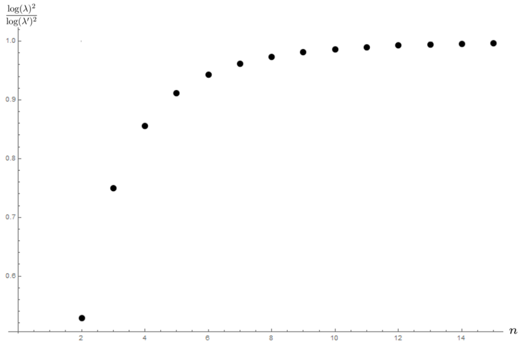
in the case that all initial energy is converted into black hole mass at late time. The quotation marks indicate that this situation is generally too naive, hence the expression above should not be taken too seriously. This picture is misleading as generally not all energy is converted to black hole mass. If sufficient energy is present to form a minimal black hole, it is not necessarily true that thermalization has to take over at late time [125].

Colliding n conical defects

The set of states for which analysis presented in the previous sections is supposed to hold is very broad. In this section a short numerical study for a subset of such states is presented. This class of states falls outside of the small class of states where the relevant monodromy problem is analytically tractable. We will assume identical pairs of operators that are evenly distributed in a polyhedral shape along the unit circle. The resulting discrete symmetry group on which one can act with this state fixes all accessory parameters with the exception of one that is left free. The remaining parameter is fixed by demanding trivial monodromy around a mirror pair of insertions z_i and $1/\bar{z}_i$, this mirror pair is depicted by the small red circle 5.5a. Given



(a) The setup; Some insertions evenly distributed around the unit circle.



(b) Ratio between the initial injected energy and the late time black hole mass derived from thermodynamics.

Figure 5.5: Colliding multiple conical defects.

the reflection property of the Fuchs equation, the following two monodromy problems are equivalent; either one fixes the accessory parameters such that the monodromy around a mirror pair is trivial or one fixes the accessory parameters such that any cycle around a single regular singular point yields a monodromy that falls within $SU(1,1)$. Given the reflection property of $T(z)$ it was shown in [97] that there exists a basis of solutions with the property that their ratio satisfies the same reflection property. From this, combined with the conjugacy class of M being contained within $SU(1,1)$ it was shown that the monodromy of this pair of solutions around a mirror pair is trivial. Since the monodromy is a basis independent property this concluded the proof that the monodromy around a mirror pair is trivial. For an extensive proof of this statement see [97]. This demand fully determines the irregular part of the stress tensor and hence it is a well-defined problem to compute the monodromy around the unit circle. Numerically integrating the connection field and determining its eigenvalue yields the temperature of the late-time equilibrium CFT state. In figure 5.5b the fraction $\log(\lambda)^2/\log(\lambda')^2$ is plotted as a function of the number of regularly distributed insertions on a polygon, with λ the eigenvalue of the monodromy matrix (5.13) and λ' defined as in (5.72). As the number of insertions increases (from two to fifteen) the fraction converges to one as the temperature starts approaching the value naively predicted by the Bekenstein-Hawking formula. In this approach the total energy $E \sim \frac{hn}{\sigma}$ that is injected was kept constant as the number of mirror pairs varies.

As can be seen in figure 5.5b for two colliding conical defects roughly half the initial energy is stored in boundary charges. Note that as the number of conical defects increases the amount of deficient energy decays rapidly. For the regularly distributed conical defects the total scattering momentum adds up to zero, as a result the black hole is expected to be at rest in the Fefferman-Graham gauge. Note that as we increase the number of conical defects, and hence increase the level of rotational symmetry we converge to the isotropic situation discussed in [104].

5.4 Discussion and outlook

This chapter considered probe correlators on a broad class of heavy states. In particular it considered the conditions under which the late time behavior of the vacuum block contribution of these correlators is indistinguishable from a thermal correlator. In the dual gravity picture this corresponds to the gravitational interaction of mass defects in AdS_3. These can eventually, at late time, collapses into a black hole spacetime, which is characterized by its Hawking temperature. In the CFT_2 picture this heavy state was created through insertions of a small number of heavy primary operators whose conformal dimensions scale with the central charge c, in the large c limit. The distribution of the operator on the radial plane was kept largely arbitrary.

The conformal vacuum block associated to this distribution of operators was obtained through means of uniformization. Although this technique works in principle for any finite number of operator insertions, it becomes a daunting task to explicitly find the uniformizing coordinates as the number of operators increases. In order to obtain a thermal collapse state dual to a black hole, one needs at least four heavy operators to create the initial heavy state. This technical obstacle is related to the fact that in order to obtain the uniformization coordinates, one needs to solve a differential equation of Fuchsian class. For four or more insertions these equations do not possess known explicit analytic solutions (except for some very specific choices of parameters which do not fit our particular needs [32]).

The main statement of this chapter is that in order to extract the semi-classical thermal behavior of correlators at late Lorentzian time, one can actually bypass solving the uniformization equation explicitly with non-trivial stress tensor expectation value. We note that the thermal behavior at late time is entirely controlled by the eigenvalues of the monodromy matrix along a curve on the unit circle. In this work we show that we can extract these eigenvalues even without the need to compute the full monodromy matrix for the solutions of Fuchs equation. The explicit relationship between the dominant eigenvalue and the temperature of the final thermal state at late Lorentzian time was presented. In the dual gravity picture this temperature corresponds to the Hawking temperature of the black hole formed through collisions among mass defects present in the bulk.

Two different routes were taken to arrive at the aforementioned connection. The first argument relies on finding a conformal transformation that preserves the structure of the monodromy matrix while leading to a new Fuchs equation with a stress tensor which can be solved analytically. For our Fuchsian system, we indeed found that one can construct an element of the orientation preserving diffeomorphism, $\mathrm{Diff}_0(S^1)$ on the circle such that the new Fuchs equation we get is associated with a primary representative. In this form the Fuchs equation can be easily solved. However, since the diffeomorphism, $\mathrm{Diff}_0(S^1)$ preserves the structure of the monodromy matrix, one can read off the required eigenvalue by solving the resulting much simpler equation. The second argument exploits the representation of the monodromy matrix in terms of a path-ordered integral over a flat connection. Through the Chern-Simons formulation of pure gravity we relate this path ordered integral to the black hole surface area. Finally, upon using the Bekenstein-Hawking formula this results in the exact same expression for the eigenvalue.

However, having a Hawking temperature at the final state of the time evolution does not necessarily mean that one has a standard BTZ black hole at the end of the collapse process. In practice this approach generates a geometry with activated boundary gravitons. The bulk states associated with black holes dressed with gravitational hair correspond to points on the same Virasoro orbit. All these points are connected though large diffeomorphisms to a reference point identified with the bulk state describing the BTZ black hole. These are precisely the same diffeomorphisms mentioned in the first proof. The eigenvalue of the monodromy matrix is preserved on a particular Virasoro orbit. Following our identification of this eigenvalue with the temperature of the collapse state, this means all these bulk states which are related through diffeomorphisms must have the same temperature. This in turn justifies the interpretation of these states as black holes states dressed with boundary gravitons. One way to distinguish between the BTZ black hole and a state corresponding to a black hole with gravitational hair is to compute the energy of the final configuration. The deficiency in the energy between the initial state and the predicted final black hole state without any gravitational hair is indicative of the amount of energy stored in boundary graviton moeds. In this chapter we additionally presented a numerical program to demonstrate this phenomenon. We found that the energy stored in boundary gravitons can actually be fairly sizable for states obtained by a small number of heavy operator insertions. However, increasing the number of insertions seems to restore rotational symmetry leading to a stationary black hole with little boundary charge energy. One can think of the case where we converge to rotationally symmetric state as converging towards the isotropic shell state [104], hence these results are consistent with the results reported in that work.

In this chapter the primary focus was on understanding the thermalization process starting from an atypical initial state. This thermal behavior is observed for the semiclassical correlators, namely from its leading order, O(1) contribution in the expansion in $\frac{1}{c}$. It would be interesting to understand the $\frac{1}{c}$ corrections to the correlator for which one would have to go to higher order corrections of the vacuum block. The corrections along with the addition of further conformal blocks are expected to modify the periodicity of Euclidean time order by order in $\frac{1}{c}$ while maintaining the KMS condition. This expectation follows from the dual gravity picture where these $\frac{1}{c}$ corrections correspond to perturbations around the semi-classical large black hole solution. However, in [95] it was shown that this periodicity actually breaks down as a result of having another non-trivial monodromy which appears while going around the time circle. Although [95] argued that these new monodromies are unphysical and should go away with a resummation it will be interesting to understand this in our setup starting with an arbitrary initial state. It will also be worth investigating the role of boundary gravitons in the thermalization process in the next leading order of perturbative expansion. In particular, it will be interesting to understand the fate of the states corresponding to black holes with gravitational hair in higher order in $\frac{1}{c}$. Another important and perhaps the most interesting direction of study will be to understand the eventual breakdown of the perturbative $\frac{1}{c}$. This happens at the scrambling time [106, 131, 110] when the non-perturbative contributions $\sim e^{O(c)}$ become important. The associated timescale characterizes the onset of chaos [107, 111, 109, 108].

Chapter 6

Thermal transition in semi-classical CFT

Strongly coupled quantum field theories can display examples of very interesting but extremely complex chaotic phenomena. Unfortunately the amount of precise statements that can be made about systems at strong coupling is very limited, and these typically only apply to very specific models. One family of strongly coupled theories where one can make rather powerful statements is 2d conformal field theory. In particular, as we saw in the last two chapters, if one narrows it down and considers 2d CFT in the regime of large central charge. In this case the amount of analytically accessible information grows even further.

Better yet, as was argued in chapter 2, a lot of these statements are often universal, depending only on conformal symmetry, modular invariance or the presence of the identity operator in the operator spectrum with a sparse spectrum assumption, while otherwise completely independent of the microscopic details of the theory. Examples include the famous Cardy formula [83],[117] and the universal ground state entanglement entropy [93][112]. A particularly fruitful common feature of 2d CFT is the universal contribution of the vacuum representation to the Hilbert space and its state-operator dual, the identity operator. In the 't Hooft limit in higher dimensions the presence of the identity operator block ensures large-N factorization, in two dimensions the extended vacuum representation of the much larger Virasoro algebra causes the identity block to capture substantially more non-trivial information of the field interactions. This has been exploited in particular in for instance [94][104].

Of course one of the most, if not the most, exciting developments in theoretical physics of the last two decades is the AdS/CFT correspondence. Its most pressing contributions are in the form of implications to quantum gravity. Rightfully so, most of the AdS/CFT literature is devoted to formulating (or possibly defining) quantum gravity in terms of conformal field theory. But the flipside of the conjecture provides us with a lot of intuition on strongly coupled conformal field theory with large numbers of degrees of freedom in terms of semi-classical gravity. For one we expect the gravitational sector of the AdS_3 bulk dual to exhibit a phase transition in the form of black hole collapse. In the boundary CFT we hence expect that non-equilibrium states evolve at asymptotically late Lorentzian times to either evolve to an equilibrium state that is either in an integrable phase or a thermal phase.

The key word in the CFT description of black hole collapse is thermalization, it is typically said that a state thermalizes if at asymptotic late times a certain class of sufficiently 'simple' correlators on the state approach the expectation values that would have been obtained if they had been computed on a thermal state instead. The subject of this chapter will be to consider the identity block contribution to a two-point function of light probe operators on a heavy non-equilibrium state, specifically at leading order in a large central charge expansion.

Summary of the central message

The distinguishing function determining whether a state thermalizes or not is the stress tensor expectation value of the state. A family of stress tensor expectation values which are connected to each other through conformal transformations form a Virasoro coadjoint orbit, all of these distinct orbits have been classified [124, 54, 53, 52]. While the role of these orbits within the context of 3d gravity with a negative cosmological constant is known [66, 81, 69, 70, 71, 125, 78, 135], in [1] the authors established from the conformal field theory perspective that the orbit in which the stress tensor expectation value of a state is contained determines its ultimate fate, this was of course reviewed in the last chapter.

In this chapter it will be demonstrated that given a non-equilibrium state that would keep the system in an integrable phase, there exists a state associated energy scale such that one can trigger thermalization. Specifically if we prepare the system in advance in an energy eigenstate with an energy above this scale and release the non-equilibrium state on top of this eigenstate the

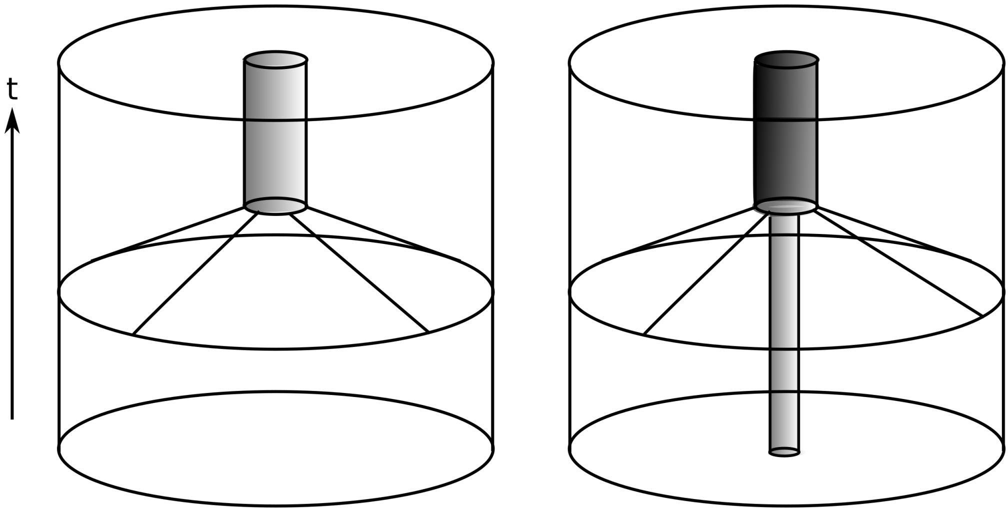


Figure 6.1: The bulk gravitational picture is particularly intuitive. If a non-equilibrium collapse state is prepared that will collapse into a non-thermal geometry (e.g. a conical defect geometry), then if the experiment were to be repeated with an initial conical defect with mass m in the asymptotic past, then there exists a critical value of m such that the conical defect is pushed over the critical opening angle and a BTZ black hole is formed instead.

system will thermalize. Furthermore it will be shown that unitarity combined with Virasoro representation theory ensures that there can only be at most one such transition scale per state.

While the gravitational bulk picture of this phenomena is very intuitive (see figure 6.1) a pure conformal field theory description of this transition for CFTs with large numbers of degrees of freedom appears to be lacking in the literature.

On a more pragmatic note, this approach naturally presents a set of lower bounds on the transition energy in terms of the other Virasoro conserved charges of the state. These provide us with a set of necessary (but not sufficient) conditions that a state needs to satisfy in order to thermalize. It is an open problem to find an analytically accessible diagnostic deciding whether a state thermalizes or not [125, 136], in principal the answer is given by a path-ordered integral but it is generally not feasible for these to be analytically computed. The necessary conditions in terms of Virasoro charges gives a partial answer to this question in terms of much simpler contour integrals instead. Alternatively, if the Virasoro conserved charges are interpreted as thermodynamic equation of state variables, then the derived lower bounds can be thought of as bounding the equation of state in the thermalizing phase.

As a mathematical problem the question: to what Virasoro coadjoint orbit does a given stress tensor belong, appears deceptively similar to a continuous version of the question: given a CFT descendent state, what is its associated primary state. While the stress tensor expectation value, in a sense, only contains coarse-grained information about a state, it is much more directly related to the thermodynamic quantities of the state. A simple example that partly demonstrates the difference between these concepts is the case of the displaced primary operator acting on the the vacuum $O(\alpha)|0\rangle = \exp(-i\alpha L_{-1})O(0)\exp(i\alpha L_{-1})|0\rangle$. Expanding the exponentials displays the fact that this simple state is a complicated superposition of descendant states at all levels of the Verma module. As this state does not have a specific level it is unclear how to ascertain purely from the Virasoro algebra what its asociated primary state is. This problem will be further exacerbated by the states under consideration in this paper as they will generically be superposition states of many different conformal families.

This chapter has been organized in the following way, section 6.1 provides a relatively low-technical review of the identity block contribution to probe two-point functions on heavy states and the role of Virasoro coadjoint orbits. Section 6.2 discusses some of the general features of Hill's equation, the class of ODEs that naturally occur in these two-point functions and presents an equivalence between a subspace of Hill's equations and the space of bounded Virasoro orbits. In section 6.3 a set of lower bounds are derived for the transition energy from an elliptic to hyperbolic state. Furthermore one appendix is relevant to this discussion. This appendix provides some of the more technical aspects of the proof for the argued equivalence in section 6.2. For a reader who is interested in an overview of the results but wishes to avoid the technical derivations reading section 6.1 is in principle sufficient.

6.1 Black hole collapse and the identity block

One particular feature of asymptotically AdS$_3$ gravity is that its black hole solutions, the BTZ black holes, possess a mass gap. As a result not all collapsing states can form black holes. Naively one could expect that the total energy of a state could act as a diagnostic to determine whether a collapse process' final state consists of a black hole. Interestingly this does not quite work due to the rich boundary dynamics attributed to the Brown-Henneaux conserved charges [18]. Simply put, arbitrarily large amounts of energy can be stored in the Virasoro boundary charges without contributing to a potential final state black hole mass, in fact one could remove or increase this energy by means of a conformal transformation [125]. As an analogy, one can consider a bound system of particles, one could increase the energy of this system by arbitrarily large amounts by boosting it to higher and higher velocities but it will not affect the Schwarzschild radius of the bound system.

By the holographic nature of asymptotic AdS$_3$ gravity it is expected that there exists a CFT analogue to this dichotomy between states $|V\rangle$ that collapse to a black hole and states that fail to form black hole states. The holographic dictionary states that the analogue would be that some initial states will equilibriate to states that are indistinguishable from being thermal [102] while other states remain in an integrable phase. The class of 2d CFTs under consideration will be very broad, mirroring the universality of black hole thermodynamics in gravitational theories. Once again the focus will be on 2d CFTs with very large central charge c, due to the semi-classical nature of black hole thermodynamics but mostly due to immense analytical simplification it provides. Secondly we will assume a sparse operator spectrum at low scaling dimensions, this allows one to find a kinematical regime where the dominant contribution to correlation functions is due to the vacuum Virasoro block, as far as the author is aware there exist no (non-problematic) examples of theories with gravitational duals that violate this assumption.

Similarly we will want to examine the late time behaviour of a set of states that is as broad as possible. One area where the large c approximation particularly shines is the calculation of correlation functions of operators with scaling dimensions that are small with respect to the central charge. Unfortunately in order to overcome the Casimir energy of the Lorentzian cylinder $|E_C| = c/24$ one needs to consider states $|V\rangle$ with scaling dimensions H_V proportional to c (typically called heavy states). The class of states that will be considered is the same as the ones of the preceeding chapter

$$|V\rangle = \prod_{k}^{N} O_k(z_k)|0\rangle, \tag{6.1}$$

with heavy operators O_k. The insertion locations z_k are not completely free since we will want to compute correlators involving this state where there exists an OPE channel where the identity block contribution dominates. One way to achieve this is to restrain z_k to a thin annulus $\mathcal{A}$ whose exterior boundary is given by the unit circle, see figure 6.2. Excluding the unit circle from

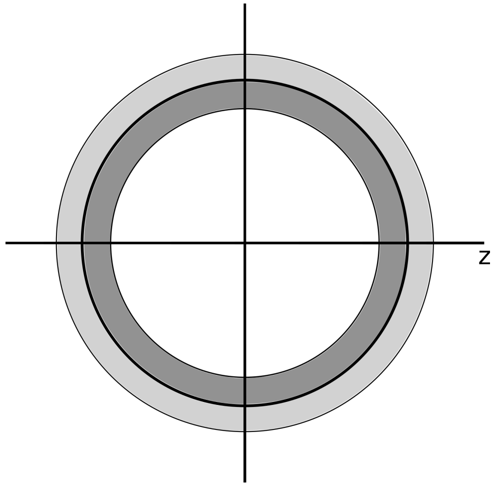


Figure 6.2: In order to ensure dominance of the identity Virasoro block the states are assumed to be generated by heavy primary operators inserted into the dark shaded annulus of width σ. As a result all hermitian adjoint operators are located in the light shaded region. The dark shaded region manifest in (6.1) as the region over which the operators are inserted.

the region $\mathcal{A}$ ensures the regularity. Note that this implies that the number of heavy operators is left generic.

The heaviness of the state provides an additional complication that was tackled by the authors of [94],[95] in what they call the uniformization approach. They applied there strategy to states generated by acting on the origin with a primary operator, in [1] the authors applied this method to a wider class of heavy states, and showed that while the resulting differential equations typically can not be solved analytically some qualitative features can be derived. It was for instance shown that if the heavy state thermalizes at late time that the temperature matches the temperature of the bulk dual black hole in the Chern-Simons formulation of AdS_3 gravity.

It will be shown that given an integrable heavy state increasing the energy of this state in a controlled way will cause it to cross a threshold that will trigger thermalization at late Lorentzian time. A state that thermalizes at late times will be referred to as a hyperbolic state, conversely states that fail to thermalize will be called elliptic states, this terminology makes particular sense in the following discussion but conveniently (though not coincidentally) this convention was already standardized in [81].

We will see that the existence of a single such threshold is fixed by a combination of representation theory of the Virasoro algebra and classic properties of linear ordinary differential equations.

6.1.1 Probe two-point function

To probe the behavior of the final state of an out-of-equilibrium state $|V\rangle$ consider the late-time limit of a two-point function of light operators on the state

$$Q(z_1, z_2) = \langle V|Q_h(z_1)Q_h(z_2)|V\rangle \tag{6.2}$$

the coordinates z are on the radial plane. The operator Q_h is light in the sense that $h \ll c$ as such the amount by which they affect the dynamics of $|V\rangle$ surmounts to a $1/c$ correction, for this reason they form appropriate probes to measure physics of the state $|V\rangle$, in order for the the correlator to be radially ordered we restrict the probe coordinates z_1, z_2 to the unit circle. Note that while this correlator is radially ordered on the Euclidean radial plane the Lorentzian correlator of interest is out of time order, this has to be taken into account when analytically continuing to the Lorentzian cylinder. The direct interest of this note is the late Lorentzian time limit, in chapter 5 it was shown that after mapping the radial plane to the Euclidean cylinder $z = \exp(\tau + i\phi)$ and analytically continuing the Euclidean time coordinate τ time-evolving in Lorentzian time corresponds to letting the probe observers undergo counter-clockwise circles on the radial plane. The non-trivial multi-valuedness of the correlator $Q(z_1, z_2)$ ensures that the result is not necessarily periodic in time.

The correlator will be simplified in two directions, only the leading order in a $1/c$ expansion will be maintained secondly only the identity block exchange is considered, i.e. we only consider the exchange of identity operators when performing OPE contractions between the operators of the state (6.1) and the probe operators. Unlike in higher dimensional CFT where the vacuum block contribution merely computes the factorized correlator in 2d CFT the vacuum block contribution to a correlation function captures highly non-trivial information between the interaction of the state onto the probes. One of the Virasoro descendants of the identity operator is the stress tensor, the AdS/CFT dictionary states that its dual field is the graviton field, therefore the bulk interpretation is that Virasoro identity block resums all graviton exchanges between the probes and the geometry generated by the heavy state.

It was found through means closely related to the uniformization of punctured Riemann surfaces [94][92] and derived in chapter 5 that the correlator

$Q(z_1, z_2)$ could be written in the form

$$Q(z_1, z_2) = \psi(z_1)^{-2h_Q} \psi(z_2)^{-2h_Q} \left(\int_{z_1}^{z_2} \psi(z)^{-2} dz \right)^{-2h_Q} \tag{6.3}$$

where $\psi(z)$ satisfies the Fuchsian ODE

$$\psi''(z) + \frac{6}{c} T(z) \psi(z) = 0, \tag{6.4}$$

and where the expectation value $T(z)$ is defined through

$$T(z) = \frac{\langle V|T(z)|V\rangle}{\langle V|V\rangle}. \tag{6.5}$$

The freedom in picking a particular basis of solutions $\psi(z)$ of (6.4) reflects the freedom in conjoining a uniformizing coordinate system with a global conformal transformation without spoiling its uniformization property. As a complex function $T(z)$ cannot be entirely generic, due to the fact that the in- and out-states are identical up to Hermitian conjugation, taking the complex conjugate of (6.5) shows that it is subject to the following Schwarz reflection property

$$T(z) = z^{-4} T(1/z^*)^*, \tag{6.6}$$

this for one implies that $z^2 T(z)$ will be real-valued along the unit circle which will be relevant when we consider the associated Hill system in section 6.2. Schwarz reflection also restrict the monodromy matrix around the unit circle M_ψ of a basis of solutions of (6.4). There has to exist a basis of solutions such that M_ψ is contained within $SU(1,1)$ [97], which restricts the eigenvalues of M_ψ in *any* basis to consist of both purely real numbers or both pure phases. Additionally the determinant constraint fixes the eigenvalues to be each others inverse $(\mu, 1/\mu)$

The late Lorentzian time limit corresponds to letting both the coordinates z_1 and z_2 perform a large number of cycles around the unit circle. If the eigenvalues are given by real numbers $(\mu, 1/\mu)$ the probe correlator is proportional to

$$Q\left(e^{2\pi i n_1} z_1, e^{2\pi i n_2} z_2\right) \propto \mu^{-2h(n_1 - n_2)} \qquad n_1 > n_2 \gg 1, \tag{6.7}$$

identifying n_i with a Lorentzian time through $t_i = 2\pi n_i$ we find that (6.7) resembles a two-point function evaluated on a thermal state with a temperature T_V given by

$$\log(|\mu|) = 2\pi^2 T_V. \tag{6.8}$$

This is a condensed version of some of the result presented in chapter 5.

6.1.2 Temperature as a conformal invariant and Virasoro orbits

The arguments reviewed above demonstrate that whether a heavy state $|V\rangle$ thermalizes and what temperature it thermalizes to is controlled by the eigenvalues of the monodromy matrix M_ψ of the independent solutions of the linear ODE (6.4). It can be shown that these eigenvalues are independent of conformal frame, consider the single-valued holomorphic mapping $w(z)$, under this transformation the stress tensor $T(z)$ transforms as

$$T(z) \to T(w) = \left(\frac{dz}{dw}\right)^2 T(z(w)) + \frac{c}{12} S[z,w] \tag{6.9}$$

it can easily be verified that if $\psi(z)$ solves the ODE (6.4) that after the substitution (6.9) the ODE (6.4) will be solved by

$$\phi(w) = \frac{1}{\sqrt{z'(w)}} \psi(z(w)). \tag{6.10}$$

Therefore the solution $\phi(w)$ under the curve $w(e^{i\theta})$ inherits the monodromy of the solution $\psi(z)$ under traversing the unit circle[1]. Hence the matrix M_ψ forms a conformal invariant of the stress tensor.

A set of functions $T(z)$ that are connected to each other through the action (6.9) is given by a Virasoro coadjoint orbit, see chapter 2 for a short review based on [124, 54, 51], for an additional review see [135]. All disconnected Virasoro coadjoint orbits have been classified, for instance the set of all hyperbolic orbits is designated by $\mathcal{B}_n(b)$. Here the index n indicates the number of zeroes of the independent solutions of (6.4) in the diagonal basis, the argument b uniquely parametrizes a certain representative element of the orbit. While all the various Virasoro coadjoint orbits have been classified (see table 6.1) not all of them are physically relevant. If we define the holomorphic contribution to the energy the usual way as the zero-mode of the stress tensor

$$E = L_0 = \frac{1}{2\pi i} \oint dz\, zT(z), \tag{6.11}$$

it is known that all orbits contain elements with arbitrarily large energies. This is not a problem but it does complicate the process of sorting $T(z)$ into an

[1] There is a small subtlety present concerning winding number, one could construct a map $w(z)$ such that a closed curve in w-space has to be traversed multiple times before $z(w)$ performs a single S^1 circle in z-space. In [124],[54] this issue is bypassed by considering only mappings $S^1 \to S^1$ with winding number 0, in this paper the issue will largely be disregarded.

Orbit	Stabilizer	$SL(2,\mathbb{R})$ class	Bounded L_0	Primary repr.
$\mathcal{B}_0(b)$	S^1	hyperbolic	yes	yes
$\mathcal{B}_{n>0}(b)$	$T_{n,\Delta}$	hyperbolic	no	no
$\mathcal{C}(\nu)$ $(0<\nu<1)$	S^1	elliptic	yes	yes
$\mathcal{C}(\nu)$ otherwise	S^1	elliptic	no	yes
$\mathcal{P}_0^+$	S^1	parabolic	yes	yes
$\mathcal{P}_1^-$	$\tilde{T}_{1,-}$	parabolic	yes	no
$\mathcal{P}_n^\pm$	$\tilde{T}_{n,\mp}$	parabolic	no	no
$\mathcal{E}_1$	$PSL(2,\mathbb{R})$	exceptional	yes	yes
$\mathcal{E}_{n\neq 1}$	$PSL^{(n)}(2,\mathbb{R})$	exceptional	no	no

Table 6.1: Summarized form of the various features of the Virasoro orbits as classified in [124] and reviewed in chapter 2.

orbit as it almost completely disconnects the temperature and the energy as quantities, in section 6.3 it will be shown that any equation of state would need to contain all other virasoro conserved charges as well. What is problematic is that some orbits possess an energy that is unbounded from below, as one of the assumptions of unitary conformal field theory is that the vacuum state is the lowest energy state in any conformal frame this leads to a contradiction, hence we can dismiss these orbits as being unphysical.

Each Virasoro orbit that possesses a lower bound on its energy also possesses a primary representative, meaning that for any $T(z)$ contained within that orbit there exists a single-valued transformation that takes it to a stress tensor of the form $T(z) = H/z^2$. This is exactly the form of the stress tensor expectation value of a state created by a primary operator with scaling weight H inserted at the origin, $O_H(0)|0\rangle$. They are related to the constant representatives of [54],[124] through the coordinate transformation $z = e^{i\theta}$, in both cases they are the representatives of an orbit whose energy saturates the lower bound. The presence of such a primary representative paints a highly non-trivial picture of the physics of semi-classical CFTs, the heavy state $|V\rangle$, being generic, will generally decompose by the operator state-correspondence into a state generated by a linear combination of operators of many different conformal families all inserted at the origin. The fact that each physical orbit contains such a primary representatives anyway has some implications with respect to the eigenstate thermalization hypothesis [102], it implies that any generic heavy state is sharply dominated by a single typical energy eigenstate, which in a CFT would be either a descendent or primary state. See appendix C for a short discussion on this point and some estimates on the growth and sharpness of the peak in the case of an heavy operator distribution corresponding to inserted mirror pairs.

The set of distinct physical orbits are dominated by the hyperbolic $\mathcal{B}_0(b)$ orbits and the elliptic $\mathcal{C}_\nu$ orbits with $0 < \nu < 1$. Besides these the physical spectrum consists of one parabolic orbit $\mathcal{P}_0^+$ and an exceptional orbit $\mathcal{E}_1$ respectively corresponding to the BTZ cross-over point and the vacuum. Each of these orbits possesses a primary representatives of the form

$$\begin{array}{ll}
\mathcal{E}_1 & T(z) = 0 \\[1em]
\mathcal{C}_\nu \ \ (0<\nu<1) & T(z) = \frac{c}{24}(1-\nu^2)\frac{1}{z^2}, \\[1em]
\mathcal{P}_0^+ & T(z) = \frac{c}{24}\frac{1}{z^2}, \\[1em]
\mathcal{B}_0(b) & T(z) = \frac{c}{24}(1+b^2)\frac{1}{z^2}.
\end{array}$$

From the AdS/CFT dictionary *mass* $\longleftrightarrow$ *scaling dimension* it can be seen that it is consistent to attribute to a hyperbolic state a bulk interpretation as being conformally related to a BTZ black hole in the center of AdS. Similarly we can think of elliptic orbits as being conformally equivalent to sourcing a conical defect geometry and of the parabolic orbit as the minimal mass BTZ geometry.

As these are the lowest energy representatives they solidify the CFT intuition that $|V\rangle$ can be thought of as being dominated by a single descendant state of Virasoro representation whose highest weight state has a scaling dimension given by the numerical factor in front of the primary representative stress tensor.

One would rightly think that it would be very difficult to couple a generic state $|V\rangle$ to the appropriate primary representative since they are related to each other by the solutions of extremely complex differential equations [124]. Fortunately there exists a large body of known theorems about ODEs closely related to (6.4). These theorems will allow us to prove that we can change the primary representative of a state by acting on a state with a primary operator at the origin, $|V\rangle_H = O_H(0)|V\rangle$. Given a set of stress tensor expectation values

$$\{T_H(z)\} = \left\{ \frac{\langle V|O_H(\infty)T(z)O_H(0)|V\rangle}{\langle V|O_H(\infty)O_H(0)|V\rangle} \ \mid \ \text{All primary } O_H \text{ in the CFT spectrum} \right\}, \tag{6.12}$$

we can think of this family as a curve through the space of all possible stress tensor expectation values parametrized by H. In section 3 it will be proven that this curve intersects a bounded Virasoro coadjoint orbit only once, by showing that the constant representative of the orbit increases monotonically as a function of H. In fact this tells us that given an elliptic state $|V\rangle$ there exists a certain state-specific critical scaling dimension H_* such that $|V\rangle_{H_*}$ falls within the class of hyperbolic states.

The bulk interpretation is that without any matter in the center of AdS an elliptic orbit $T(z)$ would source a geometry that would evolve to a non-thermal geometry (e.g. a conical defect), but if there was a point source of matter in the center of AdS before releasing $|V\rangle$ then the total scattering mass might be sufficient to trigger black hole collapse.

Finding the exact value for the energy neccessary to trigger such a transition is a very difficult problem but it is possible to determine a set of bounds on the transition energy in terms of the Virasoro conserved charges of the stress tensor. In section 4 it will be derived that for any integer $n \neq 0$ the transition value of L_0 designated $L_{|V\rangle}$ is bounded from below by

$$L_{|V\rangle} \geq \frac{c}{24}(1-n^2) + |\mathrm{Re}(L_{-n})| \tag{6.13}$$

From this one can easily interpret this instead as an infinite set of neccessary conditions for a stress tensor $T(z)$ to be contained within a hyperbolic orbit

$$L_0 > \frac{c}{24}(1-n^2) + |\mathrm{Re}(L_{-n})|, \tag{6.14}$$

to emphasize the meaning of these inequalities, *all* of these need to hold in order for a state $|V\rangle$ associated to $T(z)$ to thermalize, if just one of them fails it implies that $T(z)$ is contained within an elliptic orbit and hence $|V\rangle$ fails to thermalize.

One question that one could raise is, given that any state, including the heavy states, is decomposable into energy eigenstates and probe correlators on energy eigenstates are known [94]. Would an alternative strategy to bypass the discussion of coadjoint orbits be to decompose the heavy state $|V\rangle$ into energy eigenstates instead. The argument as why this alternative strategy would not be easier is due to the general sensitivity of a state on its decomposition coefficients. At large central charge and energies comparable to c the density of states is very large and a small change in any of the coefficients will have a very large effect on the time evolution of a state. Finding the basis decomposition coefficients of the original state to the required level of precision is a very difficult problem.

6.2 From Fuchs to Hill's equation and its implications

In order to determine the late-time fate of a state $|V\rangle$ one needs to know the monodromy of the solutions to (6.4) around the unit circle. This suggests that for the purpose of determining the monodromy, or equivalently (as we shall see) the orbit of $T(z)$, knowing the solutions on the full complex plane is largely superfluous. Therefore it makes sense to restrict ODE to the unit circle, as a result one obtains a real-valued ODE of Hill class. The class of Hill's equations are of the following general form

$$y''(x) + (\lambda + Q(x))\, y(x) = 0, \tag{6.15}$$

where the coefficient function $Q(x)$ is a periodic function satisfying

$$Q(x+\pi) = Q(x), \qquad \int_0^\pi dx\, Q(x) = 0. \tag{6.16}$$

Furthermore it is assumed that $Q(x)$ is a real function on the interval $[0,\pi]$. There exists a very substantial library of known theorems on the qualitative behavior of this large class of ODEs [132] which will be exploited in the upcoming sections. Turning our attention back to the Fuchs equation

$$\psi''(z) + \frac{6}{c} T(z)\psi(z) = 0, \tag{6.17}$$

where now $T(z)$ is complex meromorphic function subject to the reflection constraint

$$T(z) = \frac{1}{z^4} T(1/z^*)^*, \tag{6.18}$$

as mentioned before this directly implies that the product $z^2 T(z)|_{|z|=1}$ is real-valued. To rewrite this ODE into Hill form first map the unit circle to the interval $[0,\pi]$ by means of $z = e^{2ix}$, in which case the ODE transforms to

$$\psi''(x) - 2i\psi'(x) - \frac{24}{c} e^{4ix} T(x)\psi(x) = 0. \tag{6.19}$$

This can (almost) be brought into the canonical form above by means of the transformation $y(x) = e^{-ix}\psi(x)$

$$y''(x) + \left(1 - \frac{24}{c} e^{4ix} T(x)\right) y(x) = 0. \tag{6.20}$$

In order to fully connect to the canonical form (6.15) the zero-mode needs to be subtracted out of the coefficient function

$$\lambda = \frac{1}{\pi}\int_0^{\pi} dx\, 1 - \frac{24}{c}e^{4ix}T(x) = 1 - \frac{12}{\pi i c}\oint dz\, zT(z) = 1 - \frac{24}{c}L_0. \qquad (6.21)$$

Hence we obtain the Hill system (6.15) whose elements are described in table 6.2.

$y(x) = e^{-ix}\psi(x)$
$Q(x) = \frac{24}{c}\left(L_0 - e^{4ix}T(x)\right)$
$\lambda = 1 - \frac{24}{c}L_0$
$z = e^{2ix}$

Table 6.2: The constituents of the Hill system in terms of CFT data.

There are a few interesting things here to note. First that this is exclusively a well-defined Hill system if the coefficient function is guaranteed to be real for all real values of x. This in turn is guaranteed by the reflection symmetry of the stress-tensor expectation value $T(z)$ (6.18) which implied $z^2T(z)|_{|z|=1}$ is real-valued.

Secondly the number λ which plays a very important role in various stability theorems of the Hill equation, has a clear physical interpretation, being for one related to the holomorphic Euclidean energy L_0. Additionally if the total energy of the state is converted into a black hole (i.e. no boundary gravitons) λ satisfies $\sqrt{|\lambda|} \propto T_{BH}$ [1][94]. In the next section we will further expand on the CFT role of λ.

6.2.1 The off-set parameter as a CFT quantity

In a large amount of the theorems we will quote a deciding role will be played by the off-set parameter λ. Within the CFT context the parameter λ is not so esoteric as will be shown. Consider a state $|V\rangle$ and its stress tensor expectation value $T(z)$

$$T(z) = \frac{\langle V|T(z)|V\rangle}{\langle V|V\rangle}. \qquad (6.22)$$

From the state $|V\rangle$ we can construct a family of states by acting with primary operators

$$|V\rangle_H = O_H(0)|V\rangle \qquad (6.23)$$

whose holomorphic stress tensor expectation value is given by

$$T_H(z) \equiv \frac{\langle V|O_H(\infty)T(z)O_H(0)|V\rangle}{\langle V|V\rangle}. \tag{6.24}$$

Both the Fuchs and Hill equation take as input the expectation value of the stress-tensor on the Euclidean radial plane, hence in this specific context no analytic continuation to Lorentzian time is required, as a result the apparent breakdown of radial ordering is of no concern. Chapter 4 reviewed the argument that in the large c domain correlators take the general form $e^{-\frac{c}{6}f(z_i)}$ [93, 89, 50, 90], this combined with the holomorphic Ward identities fixes $T_H(z)$ to the form

$$T_H(z) = \frac{H}{z^2} + \frac{c_V}{z} + T(z) + \mathcal{O}(\sigma), \tag{6.25}$$

demanding that both $T(z)$ and $T_H(z)$ satisfy the reflection symmetry (6.6) sets the coefficient $c_V = 0$, hence

$$T_H(z) = \frac{H}{z^2} + T(z) + \mathcal{O}(\sigma). \tag{6.26}$$

The underlying idea is that to leading order in the annulus width σ the only Virasoro charge that is affected is the L_0 charge. From the table 6.2 we can as a result read off that acting with a primary operator on the state has the effect of changing the relevant Hill equation to one with the exact same potential $Q(x)$ but with a lower off-set parameter λ.

Acting with a primary operator on the state has a clear bulk interpretation, depending on the scaling dimension H it corresponds to creating in the asymptotic past either a conical defect or a black hole in the center of AdS_3. In fact one could feasibly construct an experiment in which the value of λ is varied. First prepare an ensemble of systems in various energy eigenstates then at a fixed reference create the collapse state $|V\rangle$ on top of the eigenstate by acting with the operators that create $|V\rangle$ out of the vacuum.

6.2.2 Floquet's theorem and its implications

The main advantage of rewriting the Fuchs equation into Hill form is that it provides access to the general theorems that apply to Hill's equation. Some of these classic results will be quoted without proof in this section, these have all been taken from [132] which in addition contains all the missing proofs. Some definitions are required, given a Hill system (6.15) and a set of normalized solutions $y_1(x)$ and $y_2(x)$ such that

$$y_1(0) = 1,\ y_1'(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 1. \tag{6.27}$$

We can define a characteristic equation

$$\rho^2 - (y_1(\pi) + y_2'(\pi))\,\rho + 1 = 0, \tag{6.28}$$

whose roots we denote by

$$\rho_1 = e^{i\alpha\pi},\ \rho_2 = e^{-i\alpha\pi}. \tag{6.29}$$

after these definitions we can state Floquet's theorem

Theorem 1 ***Floquet's theorem*** *If the roots ρ_1 and ρ_2 are distinct then Hill's equation has two linearly independent solutions*

$$f_1(x) = e^{i\alpha x}p_1(x),\ f_2(x) = e^{-i\alpha x}p_2(x), \tag{6.30}$$

where $p_i(x)$ are periodic functions with period π. If $\rho_1 = \rho_2$ then Hill's equation has a periodic solution with either period π (when $\rho_1 = \rho_2 = 1$) or period 2π (when $\rho_1 = \rho_2 = -1$). Let $p(x)$ denote this periodic solution and let $y(x)$ be another linearly independent solution. Then

$$y(x+\pi) = \rho_1 y(x) + \theta p(x), \tag{6.31}$$

with a constant θ. The case $\theta = 0$ corresponds to

$$y_1(\pi) + y_2(\pi) = \pm 2,\ y_2(\pi) = 0,\ y_1'(\pi) = 0. \tag{6.32}$$

The Floquet basis of solutions is equivalent to the eigenbasis of solutions discussed in section 2, this theorem is essentially a slightly more powerful version of the statement that the monodromy matrix around the unit circle of the solutions of the Fuchs equation is restricted to the set of $SU(1,1)$ matrices. It shares the same conclusion that, depending on wether (6.28) has real solutions, the ODE solutions can broadly be divided into a class of solutions that grows/shrinks exponentially and a class that stays bounded at all times. The former will be referred to as unstable solutions and the latter as stable solutions. This brings us to the second theorem, which tells us how the unstable and stable solutions are distributed over parameter space

Theorem 2 ***Oscillation theorem*** *To every differential equation* (6.15)*, there belong two monotonically increasing infinite sequences of real numbers*

$$\lambda_0, \lambda_1, \lambda_2, ...$$

and

$$\lambda'_1, \lambda'_2, ...$$

The differential equation (6.15) *possesses a solution of period* π *if and only if* $\lambda = \lambda_n$*, and a solution of period* 2π *if and only if* $\lambda = \lambda'_n$*. The sequences satisfy the inequalities*

$$\lambda_0 < \lambda'_1 \leq \lambda'_2 < \lambda_1 \leq \lambda_2 < \lambda'_3 < \lambda'_4 < \lambda_3 < \lambda_4.... \tag{6.33}$$

The solutions of (6.15) *are stable in the intervals*

$$(\lambda_0, \lambda'_1), (\lambda'_2, \lambda_1), (\lambda_2, \lambda'_3), ...$$

At the endpoints of these intervals the solutions are generally unstable. This is always true for $\lambda = \lambda_0$*. The solutions are stable for* $\lambda = \lambda_{2n+1}$ *or* $\lambda = \lambda_{2n+2}$ *if and only if* $\lambda_{2n+1} = \lambda_{2n+2}$*, and they are stable for* $\lambda = \lambda'_{2n+1}$ *or* $\lambda = \lambda'_{2n+2}$ *if and only if* $\lambda'_{2n+1} = \lambda'_{2n+2}$*.*

If we promote a basis of solutions to functions of x and λ as $y_i(x, \lambda)$ then the characteristic values solve the equations

$$y_1(\pi, \lambda_n) + y'_2(\pi, \lambda_n) = 2, \tag{6.34}$$

$$y_1(\pi, \lambda'_n) + y'_2(\pi, \lambda'_n) = -2. \tag{6.35}$$

Due to the fact that $\lambda = 1 - 24L_0/c$, the oscillation theorem is consistent with the physical intuition that if the energy is taken to infinity while keeping all the other Virasoro conserved charges fixed the state equilibriates to an unstable (black hole) phase with an arbitrarily high temperature. If we now think of the energy as a free parameter in the spirit of primary operator insertions presented in the last section then we can see that lowering the energy causes the solutions to settle down into a stable phase at some point λ_0. In principle the oscillation theorem states that instabilities can reoccur if the energy is lowered further but this possiblity will be shown to be excluded.

In order to connect the stability intervals of the oscillation theorem to the Virasoro coadjoint orbits we need one more theorem

Theorem 3 *Either all nontrivial real solutions of* (6.15) *have only a finite number of zeroes, or all real solutions have infinitely many zeroes. Let* λ_0 *be the* *smallest value of* λ *for which* (6.15) *has a periodic solution. Then for* $\lambda \leq \lambda_0$ *all nontrivial real solutions have only a finite number of zeroes, but for* $\lambda > \lambda_0$*, every real solution has infinitely many zeroes.*

This theorem hints at the fact that the zeroth instability interval is special since it is the only one where solutions possess a finite number of zeroes.

6.2.3 Equivalence to Virasoro coadjoint orbits

At this point two ways of classifying stress tensors have been discussed, one can either sort $T(z)$ into one of the Virasoro coadjoint orbits or one can try to find out in what (in)stability interval its associated Hill system falls. In this section it will be shown that both approaches are equivalent.

The basis of solutions to the Fuchs equation that diagonalizes the monodromy matrix around the unit circle corresponds to the Floquet basis of solutions of the Hill system. From this we can determine that the hyperbolic class of Fuchs equations, those with eigenvalues $|\mu_i| \neq 1$ correspond to unstable solutions of the Hill system. The set of stress tensors $T(z)$ that lead to hyperbolic Fuchs equations is given by the set of hyperbolic Virasoro coadjoint orbits $\mathcal{B}_n(b)$. The integer n counts the number of zeroes of the solutions of the Fuchs equation on the unit circle [124] in the diagonal basis, hence for the Hill system the number n corresponds to the number of zeroes on the interval $(0, \pi)$ of the Floquet basis.

Theorem 3 states that the only instability interval whose solutions can have a finite number of zeroes is the zeroth instability interval $\lambda \in (-\infty, \lambda_0)$. The solutions associated to the $\mathcal{B}_0(b)$ orbit stress tensors possess no zeroes in the Floquet basis, therefore they have to be contained within the zeroth instability interval. Conversely, all orbits $\mathcal{B}_{n>0}(b)$ lead to Hill solutions that possess at least one zero on the interval $(0, \pi)$. because of the real-valuedness of the Floquet factors ρ_i this zero has to be attributed to the periodic parts $p_i(x)$, hence on the entire real line these solutions have an infinite number of zeroes and they have to be contained within the higher instability intervals.

On physical grounds all orbits $\mathcal{B}_{n>0}(b)$ can be dismissed as they correspond to states which are conformally related to states with an energy lower than the vacuum. *This implies that all instability intervals other than the zeroth interval can be dismissed as being unphysical.*

From this we can easily establish that all stability intervals other than the first first are similarly unphysical. Simply take such a hypothetical state $|V\rangle$ such that its associated Hill system falls in one of the higher stability intervals. Then we can construct a new state $|V\rangle_H = O_H(x)|V\rangle$ by acting with a primary

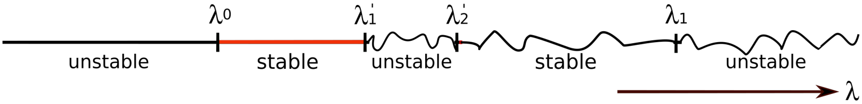


Figure 6.3: A representation of the various stability and instability intervals of Hill's equation. All intervals beyond the first instability interval can be dismissed by demanding unitarity of the CFT.

operator. As derived in section 3.1 this will only affect the Hill system by lowering the value of λ. Therefore if we are in a higher stability interval we can pick a scaling dimension H such that $|V\rangle_H$ is contained within a higher instability interval. As a result we conclude that $O_H(0)|V\rangle$ is an unphysical state and cannot be contained within the Hilbert space of our CFT which implies that $|V\rangle$ cannot be contained within the physical Hilbert space either. The remaining intervals are displayed in figure 6.3.

To complete the equivalence one needs to show the converse statement, that every point on zeroth instability interval and first stability interval respectively correspond to a unique $\mathcal{B}_0(b)$ orbit or $\mathcal{C}_{0<\nu<1}$ orbit. The proofs for these statements are quite a bit more technical and as a result have been relegated to appendix D. The underlying idea is straightforward though, there exists a simple bijective mapping of the Floquet factor α in $e^{i\alpha x}$ and the constant representative of the orbit, in appendix B it is shown that on the zeroth instability interval and the first stability interval the Floquet factor is a monotonic function of λ, which interpolates every single possible Floquet factor. As a result every single constant representative is hit exactly once on the interval $\lambda \in (-\infty, \lambda_1')$.

In short we conclude that every physical bounded Virasoro coadjoint orbit has to correspond to a point on the zeroth instability interval or the first instability interval, and conversely that every point on these two (in)stability intevals in a one-to-one manner exhausts the full set of $\mathcal{B}_0(b)$, $\mathcal{C}_{0<\nu<1}$ and the $\mathcal{P}_0^+$ orbits. This completes the proof for the proposed equivalence between physical orbits and the first two (in)stability intervals. As a direct corollary this provides a CFT version of the fact that there can only be one non-thermal to thermal gravitational phase transition in the bulk.

6.3 Putting upper bounds on the transition energy

Virasoro representation theory restricts the set of physically relevant characteristic values to λ_0 (other than λ_1' which will not be discussed). When λ_0 is converted into an energy scale $L_{|V\rangle}$ through $\lambda_0 = 1 - \frac{24}{c} L_{|V\rangle}$ it corresponds to the energy at which the family of states

$$\{|V\rangle_H\} = \{O_H(0)|V\rangle \mid \text{All primary } O_H \text{ in the CFT spectrum}\}\,, \tag{6.36}$$

crosses over to a state that eventually thermalizes at late time. Alternatively one can think of the energy scale $L_{|V\rangle}$ as a diagnostic tool that lets one easily determine whether a state $|V\rangle$ will thermalize or remain in an integrable phase. Unfortunately determining λ_0 analytically is an extremely difficult task, either given by the solution to a path-ordered integral [1], or by the solution of an infinite determinant problem [132]. In this section we will instead derive a set of upper bounds for the transition energy $L_{|V\rangle}$ in terms of the Virasoro conserved charges. For this purpose consider the following eigenvalue problem which can be constructed out of Hill's equation

$$\left(\frac{d^2}{dx^2} + \lambda + Q(x)\right) y_n(x) = \sigma_n y_n(x), \quad y_n(x+\pi) = y_n(x), \tag{6.37}$$

from the oscillation theorem we know that if $y(x)$ is constrained to have periodic boundary conditions $y(x+\pi) = y(x)$ that the eigenvalues σ_n have to be related to the characteristic values through $\sigma_n = \lambda - \lambda_n$. If we are given a stress tensor in the thermal phase we know from the arguments above that $\lambda < \lambda_0$. Hence in the thermal phase all eigenvalues are negative and

$$\mathbf{H}(x) \equiv \frac{d^2}{dx^2} + Q(x) + \lambda \tag{6.38}$$

is a negative definite operator. Assuming that the set of periodic solutions $y_n(x)$ forms a complete set on the space of periodic functions we can derive the following inequality

$$\int_0^{\pi} dx\, w(x)\mathbf{H}(x)w(x) \leq 0, \tag{6.39}$$

which should hold for any real-valued periodic function $w(x)$. We can think of $(-)\mathbf{H}(x)$ as a Hamiltonian of some quantum system in which case estimating

λ_0 corresponds to estimating the ground state energy. Note that the inequality (6.39) suggests that one possible strategy to sort $T(z)$ into an orbit would be to check whether there exists a periodic function $w(x)$ such that the inequality (6.39) fails to hold. Mind though that if $T(z)$ is contained within the elliptic phase that $\mathbf{H}(x)$ will only have one positive eigenvalue, hence trial-and-error is extremely unlikely to produce a counter-example.

By exploiting the fact that the spectrum of $\mathbf{H}(x)$ is bounded from above a more general set of bounds can be derived. This can be done by applying the variational method to estimate the ground state energy of a Hamiltonian. Since σ_0 is the largest eigenvalue of $\mathbf{H}(x)$ the following inequality has to hold for any mormalized periodic function $w(x)$

$$\int_0^\pi dx\, w(x)\mathbf{H}(x)w(x) \leq \sigma_0. \tag{6.40}$$

Take the following normalized set of test functions $w_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$.

$$\int_0^\pi dx\, w_n(x)\mathbf{H}(x)w_n(x) = \frac{1}{2}(1-n^2) - \frac{12}{c}L_0 + \frac{6}{c}\left(L_n + L_{-n}\right), \tag{6.41}$$

To obtain this expression the identities $Q + \lambda = 1 - \frac{24}{c}e^{4ix}T(x)$ and $L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z)$ have been used. Since the largest eigenvalue of $\mathbf{H}(x)$ is given by σ_0 we can establish the following general upper bound

$$1 - n^2 - \frac{24}{c}L_0 + \frac{12}{c}\left(L_n + L_{-n}\right) \leq \sigma_0, \tag{6.42}$$

reinstating λ_0 through $\sigma_0 = \lambda - \lambda_0 = 1 - \frac{24}{c}L_0 - \lambda_0$ and similarly converting λ_0 into the transition energy scale $\lambda_0 = 1 - \frac{24}{c}L_{|V\rangle}$ yields

$$\frac{c}{24}(1-n^2) + \frac{1}{2}\left(L_n - L_{-n}\right) \leq L_{|V\rangle}. \tag{6.43}$$

This provides us with an infinite set of lower bounds on the black hole transition energy. Therefore if L_0 is smaller than the left hand-side for any value of n we know for a fact that the energy of the state $|V\rangle$ is below the transition energy and we can conclude that the stress tensor has to be contained within an elliptic orbit, or to phrase it quantitatively, if there exists any n such that

$$L_0 < \frac{c}{24}(1-n^2) + \frac{1}{2}\left(L_n + L_{-n}\right) \tag{6.44}$$

then $T(z)$ has to be some element of one of the C_ν elliptic orbits.

By considering the different set of test functions $w_n(x) = \sqrt{\frac{2}{\pi}}\cos(nx)$ a similar set of lower bounds can be obtained[2]

$$\frac{c}{24}(1-n^2) - \frac{1}{2}\left(L_n - L_{-n}\right) \leq L_{|V\rangle}. \tag{6.45}$$

The interpretation of an orbit as a classical Virasoro representation suggests that the L_n charges are due to lowering operators and as such the dependence on these numbers is in a sense redundant, the reflection condition (6.18) imposes that $L_n = L^*_{-n}$ this allows us to eliminate the lowering operator charges. Furthermore the two sets of lower bounds (6.44) and (6.45) can be combined to give

$$\frac{c}{24}(1-n^2) + |\mathrm{Re}(L_{-n})| \leq L_{|V\rangle}. \tag{6.46}$$

This provides the following general necessary conditions for whether a state will thermalize

$$L_0 > \frac{c}{24}(1-n^2) + |\mathrm{Re}(L_{-n})|, \tag{6.47}$$

these are necessary in the sense that the state $|V\rangle$ will exclusively thermalize if all of these condtions hold. The major downside is of course that these are only necessary conditions not sufficient conditions. The CFT interpretation is that $|V\rangle$ is generically not a primary state but a descendent. Only the contribution to the energy from the primary contributes to the constant representative. These criteria tell us whether, after filtering out the energy contribution from a particular Virasoro raising mode, the energy is still sufficient to overcome the holomorphic Casimir energy equal to $c/24$. Alternatively one can think of the inequalities (6.47) as bounding an equation of state in the thermalizing phase if the Virasoro conserved charges are interpreted as thermodynamic equation of state variables.

6.4 Overview

The universality of black hole formation and black hole thermodynamics of gravity theories on asymptotic AdS_3 suggests that the dual description of thermalization should equally be fixed by the universal features of 2d conformal

[2]One might be concerned that this allows test functions that posses test functions with period 2π instead of π, the oscillation theorem states that Hill's equation can only possess non-trivial solutions with period 2π at the characteristic values λ'_n. Since $\lambda_0 < \lambda'_n$ for all n this does not affect the conclusion.

field theory. This chapter demonstrates that a similar transition to a thermal state is fixed by a combination of Virasoro representation theory, properties of linear differential equations and the universal vacuum block contribution to correlators. As a consequence a set of upper bounds were derived that firstly let one, in some cases, determine if a given heavy state is elliptic (i.e. non-thermal) and secondly provides lower bounds on the deficient energy required to trigger thermalization. These upper bounds are given in terms of the Virasoro charges, which are contour integrals of the stress tensor, this makes them significantly more analytically tractable than the path-ordered integral that would be required to obtain the exact answer.

These bounds possess a physical interpretation on both sides of the duality. On the gravitational side, the additional charges $L_{n<0}$ count the amount by which the excitations of the excited state $|V\rangle$ are locked up in conserved boundary charges. These locked excitations possess energy and as these charges are conserved this energy will never be able to contribute to a potential post-collapse black hole mass. As such it is not only the total energy of the initial state needs to not only overcome the holomorphic BTZ mass gap of $c/24$ but also the energy after filtering out the energy contained within boundary charges.

On the CFT side of the duality the conserved charges L_n can be interpreted as coarse-grained thermodynamic equation of state variables. If so the inequalities (6.47) can be thought of as approximating an equation of state when the system is prepared into a thermalizing phase. The fact that every physical coadjoint orbit possesses a primary representative is indicative of the statistical mechanics of 2d semi-classical CFT. As the annulus in figure 6.2 gets thinner the identity block contribution to the probe correlator grows more dominant, additionally the energy expectation value of the heavy state grows higher as well due to the fact that the heavy operators and their adjoints are brought closer together. Both the expectation value of the energy as well as the standard deviation of the energy grow as the annulus shrinks, but the expectation value grows parametrically faster than the standard deviation, as a result the energy expectation value becomes more sharply peaked the more identity block dominates OPE channel [1]. This underlies the physical explanation for the presence of the primary representative in the Virasoro coadjoint orbits, and reinforces the holographic dictionary that the statistical mechanics of large-c strongly coupled CFT corresponds to the semi-classical limit of gravity in AdS.

The states that fall in the categorization (6.1) are generally superpositions of various energy states, as such a natural question is to ask what breaks down in this construction when considering as a state a superposition of states of the form (6.1). Both in principle and practice there is no issue in analyzing these states due to the linearity of quantum mechanics, but these states are not single elements of coadjoint orbits, rather they are sums of elements of various coadjoint orbits. This follows from the wavefunction decoupling of light operators from heavy correlators [89], this decoupling ensures that the stress-tensor expectation value inherits the transformation rule of the CFT stress tensor. In the case of a superposition of states (6.1) no such common factor can be divided out of the Schwarzian ODE that appears in uniformization of Riemann surfaces and hence each term needs to be treated separately. Holographically this teaches us that heavy states of the form (6.1) source geometries that solve the Einstein field equations in the bulk, while superpositions of states of the form (6.1) would form quantum mechanical superposition geometries in the bulk.

To probe thermalization it is typical to compute correlators of a set of simple operators and see if, to leading order in a statistical mechanics large system limit, they match thermal expectation values. This paper specifically considers two-point functions of probe operators, these operators have small scaling dimensions compared to the system. Therefore they can be interpreted as corresponding to low-energy excitations in radial quantization. In the case considered the large central charge expansion plays the role of the large system limit. In principal the number of probe operators can be scaled up relatively straightforwardly as long as the resulting product operator is parametrically light compared to the central charge. Physically this correponds to the fact that a sufficiently complicated quantum measurement can resolve the microstate of the ensemble which in the process destroys the statistical mechanics interpretation. Quantitatively the Virasoro Ward identity states that one would have to sum up all light contributions to the stress tensor expecation value if there are order $O(c)$ of these $O\left(c^0\right)$ contributions the large c expansion will break down.

Intuitively black holes formed by dynamical collapse would be an ideal arena to study the information paradox. In fact since the leading order large central charge calculation reproduces the correlator decay discussed in Maldacena's formulation of the information paradox [133], it is tempting to compute perturbative corrections. Especially since the authors of [95] showed that these corrections can be computed systematically. Unfortunately almost exactly the same authors demonstrated that these perturbative corrections can not resolve the information paradox [134], see also [137] for a recent discussion. Perturbative $1/c$ corrections are not entirely devoid of practical meaning though, and are in fact very efficient for the purpose of calculating gravitational loop corrections. The first $1/c$ corrections of the probe part contributes the Eikonal resummation of ladder diagrams of graviton exchanges between the probe fields and higher order corrections resum loop corrections on top of that.

Chapter 7

Appendices

7.1 Appendix A: Four heavy operators: Heun equations

The simplest non-trivial heavy state to consider is the product of two heavy operators. As this contributes the two operators for the ket state and two additional ones for the bra state this results in a heavy sector containing four heavy operators[1]. We can ensure that they fall within the thin annulus discussed in chapters 5 and 6 by giving these heavy operators the following insertion points

$$\langle\psi(-1-\sigma)\psi(1-\sigma)\psi(-1+\sigma)\psi(1+\sigma)T(w)\rangle = 0 \tag{7.1}$$

The relevant Fuchs equation for solving the resulting uniformization problem turns into an equation of Heun type [32]

$$u''(z) + \frac{6}{c}\left(\sum_{i=1}^{4}\frac{H}{(z-z_i)^2} - \frac{b_i}{z-z_i}\right)u(z) = 0. \tag{7.2}$$

Similarly the residues b_i are defined by

$$b_i \equiv \frac{\partial}{\partial x_i}\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle \tag{7.3}$$

[1]This appendix is based on unpublished calculations performed by the author in 2016, since then a similar procedure has been reported in [136].

A relatively conservative assumption discussed at length in chapter 4 is to take the heavy correlator given above to satisfy the Zamolodchikov scaling conjecture in the semi-classical limit

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle \sim e^{-\frac{c}{6}f_0(x_i)}. \tag{7.4}$$

In this case the expression above becomes

$$u''(z) + \frac{6}{c}\left(\sum_{i=1}^{4}\frac{H}{(z-z_i)^2} + \frac{\frac{c}{6}\tilde{c}_i}{z-z_i}\right)u(z) = 0, \tag{7.5}$$

where in turn the accessory parameters are defined by

$$\tilde{c}_i \equiv \frac{\partial f_0(x_i)}{\partial x_i} \tag{7.6}$$

To make contact with the literature on the Heun equation it is useful to perform the following fractional linear transformation

$$z(x) = -\sigma\frac{x-1-\sigma}{x+1+\sigma}, \tag{7.7}$$

its purpose is to map three of our heavy operators to the canonical points $0, 1, \infty$. This transformation also effects the accessory parameters

$$\tilde{c}_i \to c_i.$$

The demand is that the full correlator is dominated by the identity block, hence to good approximation the identity block inherits the transformation rule of the full correlator, combining this with (7.4) lets one easily transform the accessory parameters to the new frame. We have access to three regularity conditions for the stress-energy tensor expectation value at infinity

$$\begin{aligned}
&\sum_{i=1} c_i = 0,\\
&\sum_{i=1}\left(c_i x_i - \frac{6h_i}{c}\right),\\
&\sum_{i=1}\left(c_i x_i^2 - \frac{12h_i}{c}x_i\right).
\end{aligned} \tag{7.8}$$

This gives us in total three constraints to fix four numbers. Solving these three linear equations gives us the following expressions for the accessory parameters, where c_1 is left as a free parameter

$$\begin{aligned} c_2 &= \frac{1}{x-1}\left(c_1 + 24\frac{H}{c}\right), \\ c_3 &= \frac{1}{1-x}\left(xc_1 + 24\frac{H}{c}\right), \\ c_4 &= 0. \end{aligned} \tag{7.9}$$

Note that we have defined $x \equiv \sigma^2$ to make contact with the notation of [114].

7.1.1 Approximating the solutions

The above Fuchs-type equation has four regular singular points, for which unfortunately no general solution is known. But the technique presented [114] allows us to approximate the solution near the regular points. In order to do this take $T(z)$ which to remind you is given by

$$T(z) = \frac{H}{z^2} + \frac{H}{(z-x)^2} + \frac{H}{(z-1)^2} + \frac{\frac{c}{6}c_1}{z} + \frac{\frac{c}{6}c_2}{z-x} + \frac{\frac{c}{6}c_3}{z-1} \tag{7.10}$$

and split it up into a part which has poles at both 0 and x and a 'correction' term which is regular at those points

$$T(z) = T_0(z) - V(z) \tag{7.11}$$

where now

$$T_0(z) = \frac{H}{z^2} + \frac{H}{(z-x)^2} - \frac{\frac{c}{6}xc_1}{z(z-x)} \tag{7.12}$$

and

$$V(z) = -\frac{H}{(z-1)^2} + \frac{c}{6}c_3\frac{1-x}{(x-z)(z-1)} \tag{7.13}$$

Note that to obtain this expression you have to apply the solutions for the accessory parameters given in 7.9. We can obtain solutions for the simplified expression

$$u''(z) + \frac{6}{c}T_0(z)u = 0. \tag{7.14}$$

It will turn out later that this ODE has a degenerate set of special solutions, to circumvent this we can regulate the wave function with some infinitesimal quantities ϵ and ξ

$$T_0^{\xi,\epsilon}(z) = \frac{H(1-\epsilon^2)}{z^2} + \frac{H(1-\xi^2)}{(z-x)^2} + \frac{\epsilon^2+\xi^2-\nu^2-1}{4z(z-x)}, \tag{7.15}$$

as a final step we will have to take limits $\epsilon, \xi \to 0$. By comparing to the Papperitz form of second order ODEs with three regular singular points we find the following equations for the exponents

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1, \\ \alpha_1\alpha_2 &= \frac{6H}{c}(1-\epsilon^2) \end{aligned} \tag{7.16}$$

and similarly

$$\begin{aligned} \beta_1 + \beta_2 &= 1, \\ \beta_1\beta_2 &= \frac{6H}{c}(1-\xi^2) \end{aligned} \tag{7.17}$$

The expression for γ_1 and γ_2 remain unchanged from the ones in [114] if we also define

$$\nu^2(x) = 4xc_1 - 1. \tag{7.18}$$

For convenience define the following function

$$f(x) = \sqrt{1 - 24\frac{H}{c}(1-x^2)}. \tag{7.19}$$

Knowing all of this we can encode the general solution of $u''(z) + T_0^{\xi,\epsilon}(z)u(z) = 0$ into the following Riemann P-symbol

$$u(z) = P\left\{\begin{matrix} 0 & 1 & \infty & \\ \frac{1+f(\epsilon)}{2} & \frac{1+f(\xi)}{2} & \frac{\nu-1}{2} & \frac{z}{x} \\ \frac{1-f(\epsilon)}{2} & \frac{1-f(\xi)}{2} & \frac{-\nu-1}{2} & \end{matrix}\right\} \tag{7.20}$$

By means of the transformation rules of the Riemann P-symbol we can reduce this to

$$u(z) = \left(\frac{z}{x}\right)^{\frac{1+f(\epsilon)}{2}}\left(1-\frac{z}{x}\right)^{\frac{1+f(\xi)}{2}} P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \frac{1+\nu+f(\epsilon)+f(\xi)}{2} & \frac{z}{x} \\ -f(\epsilon) & -f(\xi) & \frac{1-\nu+f(\epsilon)+f(\xi)}{2} & \end{matrix}\right\}$$

and similarly[2]

$$u(z) = \left(\frac{z}{x}\right)^{\frac{1-f(\epsilon)}{2}}\left(1-\frac{z}{x}\right)^{\frac{1+f(\xi)}{2}} P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \frac{1+\nu-f(\epsilon)+f(\xi)}{2} & \frac{z}{x} \\ f(\epsilon) & -f(\xi) & \frac{1-\nu-f(\epsilon)+f(\xi)}{2} & \end{matrix}\right\}$$

[2] I exchanged which one of the two α_i-exponents is called α_1

From these last two P-symbols we can read of the basis of special solutions which are regular at the point $z = 0$

$$u_\pm(z) = \left(\frac{z}{x}\right)^{\frac{1\pm f(\epsilon)}{2}} \left(1 - \frac{z}{x}\right)^{\frac{1+f(\xi)}{2}}$$
$$\times\ {}_2F_1\left(\frac{1+\nu \pm f(\epsilon) + f(\xi)}{2}, \frac{1-\nu \pm f(\epsilon)+f(\xi)}{2}; 1 \pm f(\epsilon), \frac{z}{x}\right) \quad (7.21)$$

We can find another basis of solutions which is regular at $z = x$, to accomplish this use a homographic tranformation to swap the first and second column

$$u(z) = P\begin{Bmatrix} 0 & 1 & \infty & \\ \frac{1+f(\epsilon)}{2} & \frac{1+f(\xi)}{2} & \frac{\nu-1}{2} & \frac{z}{x} \\ \frac{1-f(\epsilon)}{2} & \frac{1-f(\xi)}{2} & \frac{-\nu-1}{2} & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ \frac{1+f(\xi)}{2} & \frac{1+f(\epsilon)}{2} & \frac{\nu-1}{2} & 1-\frac{z}{x} \\ \frac{1-f(\xi)}{2} & \frac{1-f(\epsilon)}{2} & \frac{-\nu-1}{2} & \end{Bmatrix}. \quad (7.22)$$

From this we can conclude that if we follow the same steps as before we should end up with the following basis of special solutions regular at $z = x$

$$v_\pm(z) = \left(1 - \frac{z}{x}\right)^{\frac{1\pm f(\xi)}{2}} \left(\frac{z}{x}\right)^{\frac{1+f(\epsilon)}{2}}$$
$$\times\ {}_2F_1\left(\frac{1+\nu \pm f(\xi) + f(\epsilon)}{2}, \frac{1-\nu \pm f(\xi)+f(\epsilon)}{2}; 1 \pm f(\xi), 1-\frac{z}{x}\right) \quad (7.23)$$

We can find a matrix that relates these two bases of solutions by means of the following Kummer relation

$${}_2F_1(a,b;c,z) = \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c-a-b+1,1-z)$$
$$+\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b-c+1,1-z). \quad (7.24)$$

This gives us the following transformation matrix

$$\begin{pmatrix} v_+(z) \\ v_-(z) \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(-f(\epsilon))\Gamma(1+f(\xi))}{\Gamma\left(\frac{1-\nu-f(\epsilon)+f(\xi)}{2}\right)\Gamma\left(\frac{1+\nu-f(\epsilon)+f(\xi}{2}\right)} & \frac{\Gamma(f(\epsilon))\Gamma(1+f(\xi))}{\Gamma\left(\frac{1+\nu+f(\epsilon)+f(\xi)}{2}\right)\Gamma\left(\frac{1-\nu+f(\epsilon)+f(\xi}{2}\right)} \\ \frac{\Gamma(-f(\epsilon))\Gamma(1-f(\xi))}{\Gamma\left(\frac{1-\nu-f(\epsilon)-f(\xi)}{2}\right)\Gamma\left(\frac{1+\nu-f(\epsilon)-f(\xi}{2}\right)} & \frac{\Gamma(f(\epsilon))\Gamma(1-f(\xi))}{\Gamma\left(\frac{1+\nu+f(\epsilon)-f(\xi)}{2}\right)\Gamma\left(\frac{1-\nu+f(\epsilon)-f(\xi)}{2}\right)} \end{pmatrix} \begin{pmatrix} u_+(z) \\ u_-(z) \end{pmatrix}. \quad (7.25)$$

By means of the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ the basis transformation matrix (which will be called C) is given by

$$C = \begin{pmatrix} -\frac{\cos\left(\frac{\pi\nu}{2}\right)}{\sin(a\pi)} & \frac{\Gamma(a)\Gamma(1+a)}{\Gamma\left(\frac{1-v+2a}{2}\right)\Gamma\left(\frac{1+v+2a}{2}\right)} \\ \frac{\Gamma(-a)\Gamma(1-a)}{\Gamma\left(\frac{1-v-2a}{2}\right)\Gamma\left(\frac{1+v-2a}{2}\right)} & \frac{\cos\left(\frac{\pi\nu}{2}\right)}{\sin(a\pi)}) \end{pmatrix} \quad (7.26)$$

According to Mathematica the inverse is given by the following matrix

$$C^{-1} = \begin{pmatrix} -\frac{\cos\left(\frac{\pi\nu}{2}\right)}{\sin(a\pi)} & -\frac{\cos(2\pi a)+\cos(\pi\nu)}{2\sin(a\pi)^2}\frac{\Gamma\left(\frac{1-\nu-2a}{2}\right)\Gamma\left(\frac{1+\nu-2a}{2}\right)}{\Gamma(1-a)\Gamma(a)} \\ \frac{\cos(2\pi a)+\cos(\pi\nu)}{2\sin(a\pi)^2}\frac{\Gamma\left(\frac{1-\nu+2a}{2}\right)\Gamma\left(\frac{1+\nu+2a}{2}\right)}{\Gamma(1+a)\Gamma(a)} & \frac{\cos\left(\frac{\pi\nu}{2}\right)}{\sin(a\pi)} \end{pmatrix} \tag{7.27}$$

The monodromy matrices for the sets u and v around respectively 0 and x are given by

$$M_0 = M_x = \begin{pmatrix} e^{i\pi(1+a)} & 0 \\ 0 & e^{i\pi(1-a)} \end{pmatrix}, \tag{7.28}$$

where a is defined through $a \equiv \sqrt{1 - 24H/c}$. We conclude that the monodromy matrix associated to a curve enclosing 0 and x is given by

$$M_{0,x} = M_0^{-1} C M_x C^{-1} \tag{7.29}$$

the resulting matrix is quite involved so it is not so useful to display here explicitly. As discussed in chapter 4 the monodromy method applied to solving the identity block involves demanding that this matrix around the unit circle has to be conjugate to the identity matrix. This is because the operators inserted at 0 and x form a fusion pair, hence they exchange an identity operator.

We can find a generic expression for the eigenvalues of 2×2 matrix by finding the roots of the characteristic polynomial

$$\mu_\pm = \frac{1}{2}\mathrm{Tr}(M_{0,x}) \pm \frac{1}{2}\sqrt{\mathrm{Tr}(M_{0,x})^2 - 4\det(M_{0,x})} \tag{7.30}$$

The determinant is given by

$$\det(M_{0,x}) = \frac{1}{\det(M_0)} \times \det(C) \times \det(M_x) \times \frac{1}{\det(C)} = 1 \tag{7.31}$$

note that the determinant does not contain ν. Demanding the monodromy matrix to be conjugate to the identity matrix results in the constraint

$$\mathrm{Tr}(M_{0,x}) = 2. \tag{7.32}$$

This constraint can be solved analytically, for $a \in \mathbb{R}$ it is given by

$$\nu(a) = \pm\frac{1}{\pi}\arccos\left(-\frac{1 + \cos(2\pi a)\sin\left(\frac{3}{2}\pi a\right) + \sin(\pi a)\sin(2\pi a)}{1 + \sin\left(\frac{3}{2}\pi a\right)}\right), \tag{7.33}$$

the real and imaginary parts of this expression are plotted below. Note that we can see here around $a = 0.2$ our first hint appears for the emergence of a black hole threshold.

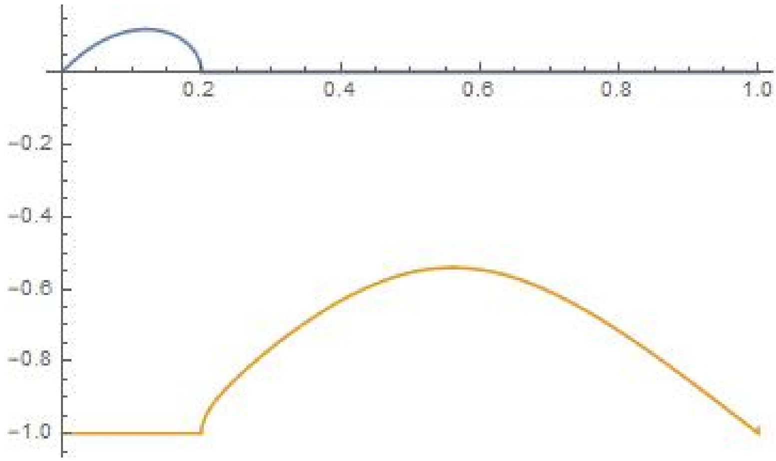


Figure 7.1: This is a plot of $\nu(a)$, the top dark curve indicates the imaginary part and the lower light line indicates the real part. Note the qualitative difference that occurs around $a = 0.2$.

7.1.2 First subleading correction to the ODE solutions

At this point it is useful to consider the first subleading correction to the approximate solution of the full Fuchs-class equation. Remember that we took the full ODE

$$u'' + \frac{6}{c}T(z)u(z) = 0, \tag{7.34}$$

and split it into the following the following one

$$u'' + \frac{6}{c}T_0(z)u(z) = \frac{6}{c}V(z)u(z). \tag{7.35}$$

Where

$$T_0(z) = \frac{H}{z^2} + \frac{H}{(z-x)^2} - \frac{\frac{c}{6}xc_1}{z(z-x)}, \tag{7.36}$$

and

$$V(z) = -\frac{H}{(z-1)^2} + \frac{c}{6}c_3\frac{1-x}{(x-z)(z-1)}. \tag{7.37}$$

We can use variation of parameters like in [89] to find an integral equation for the first correction to the solution $u^{(1)}(z)$ by solving the following inhomogenous ODE

$$\frac{d^2}{dz^2}u^{(1)} + \frac{6}{c}T(z)u^{(1)}(z) = \frac{6}{c}V(z)u^{(0)}(z) \tag{7.38}$$

The solutions to the associated homogenous equations are the leading order solutions that we already had, so we can solve this ODE by means of variation of parameters

$$u_{\pm}^{(1)}(z) = f_{1,\pm}(z)u_+(z) + f_{2,\pm}(z)u_-(z) \tag{7.39}$$

where

$$f'_{1,\pm}(z) = -\frac{6u_-(z)V(z)u_{\pm}(z)}{cW(z)}, \qquad f'_{2,\pm}(z) = \frac{6u_+(z)V(z)u_{\pm}(z)}{cW(z)}, \tag{7.40}$$

where in turn $W(z)$ is the Wronskian determinant given by

$$W(z) = u_+(z)u'_-(z) - u'_+(z)u_-(z) \tag{7.41}$$

So explicitly the first order corrections are given by

$$u_{\pm}^{(1)} = -u_+(z)\int dz\, \frac{6u_-(z)V(z)u_{\pm}(z)}{cW(z)} + u_-(z)\int dz\, \frac{6u_+(z)V(z)u_{\pm}(z)}{cW(z)}. \tag{7.42}$$

These integrals are quite awful, they form a complicated ratio of Gaussian hypergeometric functions. Numerical integration for some test values suggests that the regime of validity of our approximate solution is just a couple of multiples of σ. It seems unrealistic to demand that our solutions are valid up to $z = \frac{1}{2}$ (this is also the reason why these results are unpublished, this method fails to tell anything about the monodromy around the unit circle.)

7.2 Appendix B: From a Schwarzian equation to a Fuchsian equation

In this section it is shown how one can rewrite the Schwarzian type differential equation that was obtained by evaluating the n-point function in section 2 to a linear differential equation of Fuchsian class. We start with the highly non-linear differential equation

$$\left(\frac{dz}{dw}\right)^2 T(z(w)) - \frac{c}{12}S[z(w), w] = 0, \tag{7.43}$$

where $S[z(w), w]$ is the Schwarzian derivative defined by

$$S[z(w), w] \equiv \frac{z'''(w)}{z'(w)} - \frac{3}{2}\left(\frac{z''(w)}{z'(w)}\right)^2. \tag{7.44}$$

As a first step we invert the differential equation, so that $T(z(w))$ is a function of the variable, instead of being a variable in the equation. This is done by means of the substitution $f(z(w)) = \frac{dz(w)}{dw} = \frac{1}{w'(z)}$, where the prime denotes the derivative with respect to w. After some trivial manipulations one obtains

$$\frac{12}{c}T(z) + \frac{f''}{f} - \frac{1}{2}\frac{(f')^2}{f^2} = 0.$$

By adding and subtracting $\frac{1}{2}\frac{(f')^2}{f^2}$ we can simply replace the combination $\frac{f''}{f} - \frac{(f')^2}{f^2}$ by $\frac{d}{dz}\frac{f'(z)}{f(z)}$, this way we can obtain an equation that can be easily rewritten into a well known form

$$\frac{12}{c}T(z) + \frac{d}{dz}\frac{f'}{f} + \frac{1}{2}\frac{(f')^2}{f^2} = 0. \tag{7.45}$$

The previous expression can be simplified by defining the new function $g(z) = \frac{f(z)'}{f(z)}$. Substituting this into the expression above an equation of Riccati type is obtained,

$$g'(z) + \frac{1}{2}g(z)^2 + \frac{12}{c}T(z) = 0. \tag{7.46}$$

The Riccati type equation can be rewritten to a linear equation of Fuchsian type by defining $g(z) = 2\frac{u'(z)}{u(z)}$ and substituting we obtain

$$u''(z) + \frac{6}{c}T(z)u(z) = 0. \tag{7.47}$$

This demonstrates that the original Schwarzian equation used to define the uniformizing coordinate can be rewritten to a differential equation that is linear, which is a critical feature that will be exploited in chapters 5 and 6.

7.3 Appendix C: Energy estimates

In this section we provide some heuristic arguments to help obtain some scaling laws for the energy of the heavy states under consideration. These scaling laws provide us some intuition that we have applied in section 3 and 5 of the main body of the text. We will assume that if the separation of the mirror pairs is very small, i.e. the annulus width $\sigma \ll 1$, that the the gravitational mass of the resulting geometry is roughly the sum of the contribution of the isolated mirror pairs. The stress tensor expectation value of a single mirror pair is given by a conformal three-point function and can therefore be calculated explicitly

$$\langle O(1+\sigma)T(z)O(1-\sigma)\rangle = \frac{H}{\left((1-z)^2-\sigma^2\right)^2(2\sigma)^{2H-2}}, \tag{7.48}$$

note that we have implicitly assumed $\sigma << 1$. From this we find the expectation value

$$\langle T(z)\rangle = \frac{\langle O(1+\sigma)T(z)O(1-\sigma)}{\langle O(1+\sigma)O(1-\sigma)} = \frac{4H\sigma^2}{\left((1-z)^2-\sigma^2\right)^2}. \tag{7.49}$$

We see that the energy is peaked around $z = 1$ with a value given by $\frac{4H}{\sigma^2}$, which drops of as z^{-4} as z goes to infinity, as we expect from a stress-energy tensor expectation value. The expression above will lead to the energy density of the state created by acting with the heavy operators. The following trick can be used to obtain the energy over the full Cauchy slice. First we map to the Euclidean cylinder

$$\begin{aligned}\langle O(1+\sigma)O(1-\sigma)\rangle &= e^{H\log(1+\sigma)+H\log(1-\sigma)}\langle O(\log(1+\sigma))O(\log(1-\sigma))\rangle \\ &\sim \langle O(\sigma)O(-\sigma)\rangle,\end{aligned}$$

where we dropped all terms of order $\mathcal{O}(\sigma^2)$. After Wick rotating to Lorentzian signature

$$\langle O(i\sigma)O(-i\sigma)\rangle = \langle e^{-i(i\sigma)H}Oe^{i(i\sigma)H}e^{-i(-i\sigma)H}Oe^{i(-i\sigma)H}\rangle = \langle Oe^{-2\sigma H}O\rangle,$$

we can compute the total energy simply taking a derivative

$$\frac{\langle OHO\rangle}{\langle OO\rangle} = \frac{-\frac{1}{2}\frac{d}{d\sigma}\langle O(i\sigma)O(-i\sigma)\rangle}{\langle O(i\sigma)O(-i\sigma)\rangle} = \frac{H}{\sigma}. \tag{7.50}$$

This expression is similar to the one in [104] and consistent with the more straightforward computation in section 4.4. Note that it is only valid when σ is small. We can simply take the second-derivative to find the expectation value of the Hamiltonian squared

$$\frac{\langle OH^2O\rangle}{\langle OO\rangle} = \frac{H(2H+1)}{2\sigma^2}. \tag{7.51}$$

The standard deviation of the energy of the state is given by

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{\frac{H}{2\sigma^2}}. \tag{7.52}$$

This implies that our typical state has an energy variance that is wide enough for incorporating features like non-trivial quantum Poincaré recurrences. While $\frac{\Delta E}{E} = \frac{1}{\sqrt{2H}}$ is sufficiently small as to think of our resulting final state as a stationary black hole.

7.4 Appendix D: Floquet factor as a monotonic function of λ

The number λ contained within Hill's equation possesses within our context the interpretation as being related to the total energy of our state on the radial plane through $\lambda = 1 - \frac{24}{c}L_0$. It would hence seem sensible that decreasing λ would increase the mass of the conformal primary associated to an orbit. In this section we will prove that this is the case, in more practical terms, this establishes that as we decrease λ from λ_1' to negative infinity we sweep every physically allowed orbit and we reach every orbit exactly once. The scaling dimension H of the primary representative of an orbit $T(z) = H/z^2$ is simply related to the Floquet factor α in (6.30) through

$$\alpha = \sqrt{1 - 24\frac{H}{c}}, \tag{7.53}$$

therefore we will argue the equivalent statement that varying λ from λ_1' to negative infinity monotonically interpolates every possible Floquet factor. The proof is somewhat technical, mostly due to the fact that any proof has to naturally exclude the instability intervals that are not the zeroth interval due to the clear fact that the Floquet factor cannot be monotonic within these intervals. The proof for the first stability interval has already been done in the past, in [132] it is proven that Hill's discriminant is monotonically decreasing within this interval.

To prove the monotonicity of the Floquet factor in the zeroth instability interval consider the Floquet basis of solutions

$$y_1(x) = e^{-\beta x}p_1(x), \qquad y_2(x) = e^{\beta x}p_2(x) \qquad \text{with} \quad p_i(x+\pi) = p_i(x), \tag{7.54}$$

of Hill's equation

$$y''(x) + (Q(x) + \lambda)\, y(x) = 0 \tag{7.55}$$

an important point to note is that all solutions associated to Hill's equation with $\lambda < \lambda_0$ have only a finite amount of zeroes, the periodic nature of $p_i(x)$ hence implies that $y_i(x)$ has no zeroes. This means that both functions $y_i(x)$

have definite sign for all x and since Hill's equation is linear we can choose this sign to be positive without loss of generality for both Floquet solutions

$$y_i(x) > 0 \qquad \text{for all } x \text{ as long as } \lambda < \lambda_0. \tag{7.56}$$

now consider the following Hill equation shifted by a small ϵ

$$v''(x) + (Q(x) + \lambda - \epsilon)v(x) = 0, \tag{7.57}$$

this equation will have another Floquet basis

$$v_1(x) = e^{-\gamma x} q_1(x), \qquad v_2(x) = e^{\gamma x} q_2(x) \qquad \text{with} \quad q_i(x+\pi) = q_i(x), \tag{7.58}$$

The same consideration that applied to $y_i(x)$ apply to $v_i(x)$, i.e. the functions $v_i(x)$ can be taken to be strictly greater than zero for all x. The solutions of Hill's equation should continuously flow into each other as we vary λ. Consider the solution $\tilde{y}_2(x)$ to (7.57) obtained from the exponentially increasing $y_2(x)$ by continuously shifting $\lambda \to \lambda - \epsilon$. As a solution it should be expressible as a linear combination of Floquet solutions

$$\tilde{y}_2(x) = a v_1(x) + b v_2(x) = a e^{-\gamma x} q_1(x) + b e^{\gamma x} q_2(x), \tag{7.59}$$

since $y_2(x)$ is positive everywhere we can choose an ϵ such that the solution $\tilde{y}_2(x)$ is positive everywhere as well, this implies that both $a, b > 0$. Now consider the ratio $\tilde{y}_2(x)/y_2(x)$ in the limit $x \to \infty$

$$\lim_{x\to\infty} \frac{\tilde{y}_2(x)}{y_2(x)} = \lim_{x\to\infty} \frac{a e^{-\gamma x} q_1(x) + b e^{\gamma x} q_2(x)}{e^{\beta x} p_2(x)} = \lim_{x\to\infty} b e^{(\gamma-\beta)x} \frac{q_2(x)}{p_2(x)}. \tag{7.60}$$

Since $q_2(x)$ is a periodic function and hence bounded and since $p_2(x)$ can vanish nowhere we can conclude that if $\tilde{y}_2(x)/y_2(x)$ blows up in the limit $x \to \infty$ this implies that $\gamma > \beta$.

We will now construct the solution $\tilde{y}_2(x)$ for small ϵ by perturbing around the solution $y_2(x)$ i.e.

$$\tilde{y}_2(x) = y_2(x) + \epsilon z(x). \tag{7.61}$$

To linear order in ϵ we find that $z(x)$ satisfies the ODE

$$z''(x) + (Q(x) + \lambda)z(x) = y_2(x), \tag{7.62}$$

by means of variation of parameters we find that the relevant particular solution for $z(x)$ is given by

$$\begin{aligned} z(x) &= y_2(x) \int_0^x y_1(x') y_2(x') dx' - y_1(x) \int_0^x y_2(x')^2 dx' \\ &= e^{\beta x} p_2(x) \int_0^x p_1(x') p_2(x') dx' - e^{-\beta x} p_1(x) \int_0^x e^{2\beta x'} p_2^2(x') dx'. \end{aligned} \tag{7.63}$$

Due to the periodic nature of $p_2(x)$ it is bounded from above, take M to be an upper bound such that for all x $p_2(x) < M$ then

$$\begin{aligned} z(x) &> e^{\beta x} p_2(x) \int_0^x p_1(x') p_2(x') dx' - e^{-\beta x} p_1(x) \int_0^x e^{2\beta x'} M^2 dx' \\ &= e^{\beta x} p_2(x) \int_0^x p_1(x') p_2(x') dx' - e^{-\beta x} p_1(x) M^2 \left(\frac{1}{2\beta} e^{2\beta x} - 1 \right) \\ &> e^{\beta x} \left(p_2(x) \int_0^x p_1(x') p_2(x') dx' - p_1(x) \frac{M^2}{2\beta} \right), \end{aligned} \tag{7.64}$$

The negative term in parenthesis is bounded whereas the positive term diverges linearly with x in the limit $x \to \infty$, again due to the positive definite and periodic nature of $p_i(x)$. Looking back at the ratio $\tilde{y}_2(x)/y_2(x)$ in the large x limit

$$\begin{aligned} \lim_{x\to\infty} \frac{\tilde{y}_2(x)}{y_2(x)} &= \lim_{x\to\infty} \frac{y_2(x) + \epsilon z(x) + O\left(\epsilon^2\right)}{y_2(x)} \\ &= \lim_{x\to\infty} 1 + \epsilon \left(\int_0^x p_1(x') p_2(x') dx' - \frac{M^2}{2\beta} \frac{p_1(x)}{p_2(x)} \right) + O(\epsilon^2), \end{aligned} \tag{7.65}$$

this ratio diverges to positive infinity in the limit $x \to \infty$, from which we can establish that $\gamma > \beta$. This completes the proof that the Floquet factor is a stricly decreasing function of λ as long as $\lambda < \lambda_0$.

Chapter 8

List of publications

This thesis is based on:

- G. Vos, “Vacuum block thermalization in semi-classical 2d CFT,” JHEP **1902**, 022 (2019) doi:10.1007/JHEP02(2019)022 [arXiv:1810.03630 [hep-th]
- S. Banerjee, J. W. Brijan and G. Vos, “On the universality of late-time correlators in semi-classical 2d CFTs,” JHEP **1808**, 047 (2018) doi:10.1007/JHEP08(2018)047 [arXiv:1805.06464 [hep-th]]

Other publications from the author include:

- M. Baggio, V. Niarchos, K. Papadodimas, and G. Vos, “Large-N correlation functions in N = 2 superconformal QCD,” JHEP **01** (2017) 101, arXiv:1610.07612 [hep-th].
- G. Vos, “Generalized Additivity in Unitary Conformal Field Theories,” Nucl. Phys. B **899**, 91 (2015). [arXiv:1411.7941 [hep-th]].

Chapter 9

Samenvatting

Door het werk begonnen door Jacob Bekenstein, Stephen Hawking en anderen zijn we bewust geworden van het feit dat een kwantum beschrijving van zwaartekracht meer is dan slechts een academische uitdaging. De semiklassieke kwantum eigenschappen van zwarte gaten tonen aan dat zelfs op de schalen waar wij aan gewend zijn mogelijk grote inconsistenties kunnen bestaan tussen de basisbeginselen van kwantummechanica en semiklassieke zwaartekracht.

De ontwikkeling van het holografisch principe schijnt tot zekere hoogte meer licht op deze kwestie. Dit principe stelt dat gegeven het universum als een grote cilinder, dat de volledige kwantum mechanische en gravitationele dynamica die plaatsvind in deze cilinder volledig bevat is in een kwantumsysteem dat gebonden is aan de rand van deze cilinder. Als het waar is dat de *volledige* dynamica bevat is in dit randmodel dan zou dat onder andere impliceren dat in het binnenste van de cilinder het principe van unitariteit stand zou moeten blijven houden.

Dit holografisch principe komt in zijn meest expliciete vorm naar boven in de AdS/CFT correspondentie van Juan Maldacena. In deze constructie wordt onder andere verondersteld dat de super-Yang-Mills eiktheorie met 4 superladingen overeen komt met type IIB snaartheorie op het product van een vijf-dimensionale anti-de Sitter ruimte en een vijf-dimensionale bol. De veronderstelling daarentegen bestaat dat het holografisch principe een algemene eigenschap van kwantumzwaartekracht systemen is.

De meest vooruitspringende eigenschap van $\mathcal{N} = 4$ SYM is dat het een hoeksgetrouwe veldentheorie is (ofwel CFT, conformal field theory). Het is een interessant vraagstuk om te beschouwen in hoeverre mate de emergentie van een zwaartekracht theorie vanuit een CFT een algemene eigenschap van deze klasse van veldentheoriën is. Dit is een vraagstuk dat met name in twee-dimensionale CFTs tot zijn recht komt. Aan deze modellen kan een bepaald getal, de centrale lading c, toegekent worden. Het blijkt dat bijna zonder verdere aannames deze modellen extreem versimpelen in het regime waar dit getal c erg groot is.

Deze klasse van zogenaamde semi-klassieke CFTs bevindt zich op het evenwichtspunt tussen modellen die simpel genoeg zijn om analytische conclusies te kunnen trekken maar gecompliceerd genoeg dat ze interessante fenomenen kunnen omvatten. Vooral het fenomeen thermalizatie speelt een kritieke rol in dit proefschrift. De klasse van modellen die beschouwd worden is erg breed maar niet alomvattend, de volgende minimalistische lijst van aannames werd aangedragen:

- Unitariteit,
- Invariantie onder hoeksgetrouwe transformaties,
- Een grote centrale lading,
- Een scheiding in de schalingsdimensie tussen de identeitsoperator en de eerstvolgende primaire operator,
- een schalings aanname opgelegt aan correlatie functies van operatoren met grote schaaldimensies.

Wat semi-klassieke CFTS toegankelijk houdt is de diepe connectie tussen deze modellen en de uniformizatie van Riemann oppervlakten met puntdefekten. De hoeksgetrouwe transformaties die deze Riemann oppervlakten naar de Poincaré schijf brengen hebben de CFT interpretatie als wel transformaties naar geometriën die de aanwezigheid van operatoren met grote schaaldimensies afschermen.

De transformatie die aan deze eigenschap voldoet moet een oplossing zijn van bepaalde differentiaal vergelijkingen. De oplossingen van deze differentiaal vergelijkingen zijn functies die over het algemeen meervoudige waarden aanleveren. De monodromiën die deze oplossingen hebben om de eenheidscirkel op het complexe vlak levert informatie op over of een kwantumtoestand wel of niet thermodynamisch evenwicht zal bereiken. Als deze toestanden dat wel doen dan zal, als geconcludeerd in hoofdstuk 5, deze temperatuur overeen komen met die van Bekenstein-Hawking temperatuur van de geometrie van het zwaartekracht model in de binnenkant van de cilinder.

Dit is een conclusie die aangeeft dat het we wel degelijk hoopvol kunnen zijn dat deze semi-klassieke CFTs in hun algemeen een holografisch duaal hebben. Op basis hiervan werd gekeken of er nog meer bekende fenomenen binnen zwaartekracht een equivalent binnen de CFT vertonen. In hoofdstuk 6 werd geconcludeert op basis van de bekende eigenschappen van periodische differentiaal vergelijkingen en Virasoro coadjuncte banen dat CFTs een fase-overgang van een ergodische naar een thermische fase vertonen. Deze fase-overgang bezit een intuïtieve interpretatie aan de zwaartekracht zijde. Door het bekende feit dat er een minimum ligt aan de massa die een zwart gat kan hebben in drie-dimensionale anti-de Sitter ruimte is het mogelijk dat een hoge-energie verstrooiingsexperiment er van af kan zien om een zwart gat te produceren. Dit kan verholpen worden door het experiment te herhalen op een achtergrondgeometrie die overeen komt met anti-de Sitter ruimte met een zware puntmassa in het midden. Als nu het verstrooiingsexperiment herhaald wordt op deze achtergrond kan het resultaat wel een zwart gat opleveren als de puntmassa in het midden zwaar genoeg is. Dit precieze fenomeen wordt weerspiegeld door de CFT, verder werd er op basis van deze conclusies bepaald diagnostisch gereedschap afgeleid dat iemand in principe op analytische wijze kan vertellen of een CFT kwantumtoestand zal thermalizeren of niet. Het is een open vraag of er een manier is om van een semi-klassieke kwantumtoestand af te lezen of deze zal thermalizeren of niet. de diagnostiek gepresenteerd in hoofdstuk beantwoord in dele deze vraag.

Dit levert een bijdrage aan de veronderstelling binnen de theoretische natuurkunde gemeenschap op dat de thermodynamica van semi-klassieke CFTs overeen komt met het klassieke deel van het zwaartekracht model aan de binnenkant.

Chapter 10

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